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Combining of Jordan Algebras and Lie Groups

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Abstract

In this work, we discuss Jordan algebras connected to Lie groups, as well as how they relate to Lie algebras. As an application, we study Jordan algebra of the special unitary groups, special linear algebra and special orthogonal groups.

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1 Introduction

Let $\mathcal{B}$ be a vector space over a field $\mathbb{F}$, let $a,\ b \in \mathcal{B}$, then $\mathcal{B}$ is a Jordan algebra with product $\{a,\ b\} = ab + ba$. The vector space $\mathcal{B}$ with the commutator $[a,\ b] = ab - ba$ is a Lie algebra. If *char* $\mathbb{F} = 2$, meaning that $1 + 1 = 0$ and $1 = -1$, then we can say that the Lie algebra equal to the Jordan algebra $(ab + ba = ab - ba)$. The paper at hand considers Lie and Jordan algebra that have finite dimensions over fields $\mathbb{R}$ or $\mathbb{C}$ of characteristic zero. The fundamental goal of this research is to keep using Lie groups and Jordan algebras to advance these sciences. The following question serves as a starting point:

• Is it possible to link the Jordan algebra to the Lie group to facilitate the

application of definitions and theorems to numbers to be clearer and more accurate in understanding?
Previous research has indicated that the Lie algebra is simply the identities tangent space with a Lie bracket. We know that the Lie triple system $\mathcal{A}$ is a generalization of the Lie algebra. There exists the relation between Lie and Jordan triple system:

$$[a, b, c] = ab^t c + cb^t a - ba^t c + ca^t b,\ \forall\ a,\ b,\ c \in \mathcal{A}.$$

The main results in this paper are:

1. Jordan algebras to Lie groups are the same as Lie algebras of Lie groups, with the difference in the product commutator.

2. Giving several examples on Jordan algebras taken from Lie algebras in addition to detailing in the special unitary group $\mathfrak{su}(3)$.

The work was broken into the following sections: In Section 1 we mention some definitions relevant to our work. Our results, we study in detail in the second section.

2 Lie groups and Jordan algebras

In this section, we review several facts and concepts of Lie groups and Jordan algebras. The definition of a Lie group is given below.

Definition 2.1. *A Lie group G is a C^∞ manifold and G is a group such that:*

1. $\pi : G \times G \longrightarrow G\ ; (g, h) \longmapsto gh$

2. $inv : G \longrightarrow G; x \longmapsto x^{-1}$

π and inv are both C^∞.

Lie and Jordan algebras are not associative. A definition of Jordan algebra is as follows:

Definition 2.2. *A Jordan algebra is an algebra A^+ over a field $\mathbb{F}$, whose product meets the axioms listed below. For all $a,\ b \in A^+$ we have:*

1. $\{a, b\} = \{b, a\}$, *(commutativity).*

2. $\{\{a^2, b\}, a\} = \{a^2, \{b, a\}\}$, *(Jordan identity).*

Definition 2.3. *A vector space M over a field $\mathbb{F}$ which endowed with a trilinear mapping $\{*,\ *,\ *\}$: $M \times M \times M \longrightarrow M$. If the following requirements are met, the system is a Jordan triple system: For all $u,\ v,\ w,\ x,\ y \in M$*

1. $\{u,\ v,\ w\} = \{w,\ v,\ u\}$.
2. $\{x,\ y,\ \{u,\ v,\ w\}\} - \{u,\ v,\ \{x,\ y,\ w\}\} = \{\{x,\ y,\ u\},\ v,\ w\} - \{u,\ \{y,\ x,\ v\},\ w\}$.

Lemma 2.1. *The Jordan triple system M with respect to the product*

$$\{b_1,\ b_2,\ b_3\} = b_1 b_2^t b_3 + b_3 b_2^t b_1.$$

Where b_2^t denotes the transpose matrix of b_2 and $b_1,\ b_2,\ b_3 \in M$.

In the following, we recall the Lie algebra of a matrix Lie group and give an example of that:

Theorem 2.1. *Let g be a topologically closed subgroup of $GL(n,\ \mathbb{F})$ define*

$$g = \left\{x \in \mathrm{GL}(n,\ \mathbb{F}) : e^{tx} \in G,\ \forall\ t \in \mathbb{R}\right\},$$

where e^x the exponential function. Then:

1. *g is a vector space.*
2. *For all $x,\ y \in g \Rightarrow [x,\ y] = xy - yx \in g$.*
3. *g is parallel to tangent space of G at I.*
4. *$\exp : g \to G$ is locally invertible.*

Example 2.1. *The effect of the special linear group using the exponential function.*
We have:

$$\exp : gl(n,\ \mathbb{R}) \to GL(n,\ \mathbb{R})$$

$$SL(2,\ \mathbb{C}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : \det \exp t \begin{pmatrix} x & y \\ z & w \end{pmatrix} = 1 \right\}$$

$$\begin{aligned} \det \exp t \begin{pmatrix} x & y \\ z & w \end{pmatrix} &= \det \exp \begin{pmatrix} 1+tx & ty \\ tz & 1+tw \end{pmatrix} \\ &= (1+tx)(1+tw) - t^2 yz \\ &= 1 + t(x+w) = 1,\ \forall\ t \end{aligned}$$

$$\left.\frac{d}{dt}\right|_{t=0} det \exp t \begin{pmatrix} x & y \\ z & w \end{pmatrix} = x + w = 0.$$

Hence, $\mathfrak{sl}(2,\ \mathbb{C}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : \mathrm{Trace} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = 0 \right\}$.

3 Main Results

We aim in this section to consider the relation between Lie groups of Lie algebras and link them with Jordan algebras.

Theorem 3.1. *The Lie algebra* $\mathfrak{sl}(2,\ \mathbb{C})$ *is a Jordan algebra.*

Proof. We know that the basis for $\mathfrak{sl}(2,\ \mathbb{C})$ are

$$e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_{1(-1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now, we study the conditions of Jordan algebra: If $a = e_{1(-1)}$ and $b = e_{21}$ in Definition 2.2, then

1. $$\{e_{1(-1)}, e_{21}\} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \{e_{21}, e_{1(-1)}\}$$

2. $$\{\{e^2_{1(-1)}, e_{21}\}, e_{1(-1)}\} = \{e^2_{1(-1)}, \{e_{21}, e_{1(-1)}\}$$
$$\left\{\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_{21}\}, e_{1(-1)}\right\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \{e_{21}, e_{1(-1)}\}\right\}$$
$$\left\{\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, e_{1(-1)}\}\right\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Therefore, $\mathfrak{sl}(2,\ \mathbb{C})$ is the Jordan algebra.

□

Theorem 3.2. *The Lie algebra* $\mathfrak{so}(3,\ \mathbb{R})$ *is a Jordan algebra.*

Proof. We have:

$$\mathfrak{so}(3,\ \mathbb{R}) = \{f : f \in \mathfrak{gl}(3,\ \mathbb{R}) \text{ such that } f^t = -f\}$$

The basis of $\mathfrak{so}(3,\ \mathbb{R})$ are:

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ f_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } f_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we take $a = f_1$ and $b = f_2$ in Definition 2.2, then

1. $\{f_1, f_2\} = \{f_2, f_1\}$, where $\{f_1, f_2\} = f_1 f_2 + f_2 f_1$.

2. $\{\{f_1^2, f_2\}, f_1\} = \{f_1^2, \{f_2, f_1\}\}$.

Thus, the conditions are fulfilled. So, $\mathfrak{so}(3,\ \mathbb{R})$ is a Jordan algebra.

$\square$

The following theorem be demonstrated the special unitary group of dimension 3×3 in detail.

Theorem 3.3. *The Lie algebra* $\mathfrak{su}(3,\ \mathbb{C})$ *is a Jordan algebra and Jordan triple system.*

Proof. Let us specify what kind of groups $U(n)$ and $SU(n)$:

1. The unitary group is described as follows:

$$U(n) = \{U \in Mat(n, \mathbb{C}) : \bar{U}^t U = U \bar{U}^t = I_n,\ \text{such that } \bar{U}^t = U^{-1}\}.$$

2. The special unitary group is described as follows:

$$SU(n) = \{U \in U(n) : det(U) = 1\}.$$

If $n = 3,\ \ SU(3) = \{U \in U(3) : det(U) = 1\}$. The number of basis of $SU(3)$ equal to $n^2 - 1 = 8$. We study the Jordan algebra and Jordan triple system of $SU(3)$ as follows: the basis of a Gell-Mann matrices are:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

These matrices are the basis of $\mathfrak{su}(3)$. The Jordan algebra conditions are:

(i) The first condition is discussed in detail.

(ii) For example, if we take, $\lambda_1,\ \lambda_2$, then $\{\{\lambda_1^2,\lambda_2\},\lambda_1\} = \{\lambda_1^2,\{\lambda_2,\lambda_1\}\}$. The first condition of Jordan algebra of $\mathfrak{su}(3)$ as follows: if $i=j$, then

$$\{\lambda_1,\lambda_1\} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \{\lambda_2,\lambda_2\} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \{\lambda_3,\lambda_3\} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_4,\lambda_4\} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \{\lambda_5,\lambda_5\} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \{\lambda_6,\lambda_6\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\{\lambda_7,\lambda_7\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \{\lambda_8,\lambda_8\} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{pmatrix}$$

Therefore, if $i=j$, $\{\lambda_i,\lambda_j\} = \frac{4}{3}I_3 + \frac{1}{2}d_{ijk}\lambda_8$. Where d_{ijk} equal:

$$\{\lambda_1,\lambda_1\} = \frac{4}{3}I_3 + \frac{1}{2}\frac{1}{\sqrt{3}}\lambda_8, \quad \{\lambda_2,\lambda_2\} = \frac{4}{3}I_3 + \frac{1}{2}\frac{1}{\sqrt{3}}\lambda_8$$
$$\{\lambda_3,\lambda_3\} = \frac{4}{3}I_3 + \frac{1}{2}\frac{1}{\sqrt{3}}\lambda_8, \quad \{\lambda_4,\lambda_4\} = \frac{4}{3}I_3 + \frac{1}{2}(-\frac{1}{2\sqrt{3}})\lambda_8$$
$$\{\lambda_5,\lambda_5\} = \frac{4}{3}I_3 + \frac{1}{2}(-\frac{1}{2\sqrt{2}})\lambda_8, \quad \{\lambda_6,\lambda_6\} = \frac{4}{3}I_3 + \frac{1}{2}(-\frac{1}{2\sqrt{3}})\lambda_8$$
$$\{\lambda_7,\lambda_7\} = \frac{4}{3}I_3 + \frac{1}{2}(-\frac{1}{2\sqrt{3}})\lambda_8, \quad \{\lambda_8,\lambda_8\} = \frac{4}{3}I_3 + \frac{1}{2}(-\frac{1}{\sqrt{3}})\lambda_8$$

On the other hand, if $i \neq j$, then

$$\{\lambda_1,\lambda_2\} = 0 = \{\lambda_2,\lambda_1\} \qquad \{\lambda_1,\lambda_3\} = 0 = \{\lambda_3,\lambda_1\} \qquad \{\lambda_1,\lambda_4\} = \lambda_6 = \{\lambda_4,\lambda_1\}$$
$$\{\lambda_1,\lambda_5\} = \lambda_7 = \{\lambda_5,\lambda_1\} \qquad \{\lambda_1,\lambda_6\} = \lambda_4 = \{\lambda_6,\lambda_1\} \qquad \{\lambda_1,\lambda_7\} = \lambda_5 = \{\lambda_7,\lambda_1\}$$
$$\{\lambda_1,\lambda_8\} = \frac{2\sqrt{3}}{3}\lambda_1 = \{\lambda_8,\lambda_1\}.$$

$$\{\lambda_2,\lambda_3\} = 0 = \{\lambda_3,\lambda_2\} \qquad \{\lambda_2,\lambda_4\} = -\lambda_7 = \{\lambda_4,\lambda_2\} \qquad \{\lambda_2,\lambda_5\} = \lambda_6 = \{\lambda_5,\lambda_2\}$$
$$\{\lambda_2,\lambda_6\} = \lambda_5 = \{\lambda_6,\lambda_2\} \qquad \{\lambda_2,\lambda_7\} = -\lambda_4 = \{\lambda_7,\lambda_2\} \qquad \{\lambda_2,\lambda_8\} = \frac{2\sqrt{3}}{3}\lambda_2 = \{\lambda_8,\lambda_2\}.$$

$$\{\lambda_3,\lambda_4\} = \lambda_4 = \{\lambda_4,\lambda_3\} \qquad \{\lambda_3,\lambda_5\} = \lambda_5 = \{\lambda_5,\lambda_3\} \qquad \{\lambda_3,\lambda_6\} = -\lambda_6 = \{\lambda_6,\lambda_3\}$$
$$\{\lambda_3,\lambda_7\} = -\lambda_7 = \{\lambda_7,\lambda_3\} \qquad \{\lambda_3,\lambda_8\} = \frac{2\sqrt{3}}{3}\lambda_3 = \{\lambda_8,\lambda_3\}.$$

$\{\lambda_4, \lambda_5\} = 0 = \{\lambda_5, \lambda_4\}$ $\quad \{\lambda_4, \lambda_6\} = \lambda_1 = \{\lambda_6, \lambda_4\}$ $\quad \{\lambda_4, \lambda_7\} = -\lambda_2 = \{\lambda_7, \lambda_4\}$

$\{\lambda_4, \lambda_8\} = -\frac{\sqrt{3}}{3}\lambda_4 = \{\lambda_8, \lambda_4\}.$

$\{\lambda_5, \lambda_6\} = \lambda_2 = \{\lambda_6, \lambda_5\}$ $\quad \{\lambda_5, \lambda_7\} = \lambda_1 = \{\lambda_7, \lambda_5\}$ $\quad \{\lambda_5, \lambda_8\} = -\frac{\sqrt{3}}{3}\lambda_5 = \{\lambda_8, \lambda_5\}.$

$\{\lambda_6, \lambda_7\} = 0 = \{\lambda_7, \lambda_6\}$ $\quad \{\lambda_6, \lambda_8\} = -\frac{\sqrt{3}}{3}\lambda_6 = \{\lambda_8, \lambda_6\}.$

$\{\lambda_7, \lambda_8\} = -\frac{\sqrt{3}}{3}\lambda_7 = \{\lambda_8, \lambda_7\}.$

Hence, if $i \neq j$, then $\{\lambda_i, \lambda_j\} = a_i\epsilon_{ijk}\lambda_k$. Where $a_i = \frac{\sqrt{3}}{3}$ or $\frac{2\sqrt{3}}{3}$. For all $i, j = 1, 2, 3, 4, 5, 6, 7$ and $\{\lambda_8, \lambda_i\} = a_i\epsilon_{ijk}\lambda_i$.

Now, we study the Jordan triple system of $\mathfrak{su}(3)$: the conditions of the Jordan triple system as true of $\mathfrak{su}(3)$. The case if $i = j = k$, then

$\{\lambda_1, \lambda_1, \lambda_1\} = 2\lambda_1$ $\;\{\lambda_2, \lambda_2, \lambda_2\} = -2\lambda_2$ $\;\{\lambda_3, \lambda_3, \lambda_3\} = 2\lambda_3$ $\;\{\lambda_4, \lambda_4, \lambda_4\} = 2\lambda_4$

$\{\lambda_5, \lambda_5, \lambda_5\} = -2\lambda_5$ $\;\{\lambda_6, \lambda_6, \lambda_6\} = 2\lambda_6$ $\;\{\lambda_7, \lambda_7, \lambda_7\} = -2\lambda_7$ $\;\{\lambda_8, \lambda_8, \lambda_8\}$

$= 2(\frac{\sqrt{3}}{9})\lambda_8.$

So, if $i = j = k$, $\{\lambda_i, \lambda_i, \lambda_i\} = 2\epsilon_{ijk}a_i\lambda_i$. The case $i \neq j \neq k$ it is achieved similarly, as in the case of the Jordan algebra, with the difference in the process of multiplying the bracket set. Consequently, the algebra $\mathfrak{su}(3)$ is the Jordan algebra and Jordan triple system.

□

Thus, from theorems on this section: we can say that

Corollary 3.4. *The Jordan algebra relationship with Lie group is the same as the relationship of the Lie algebra of the Lie group.*

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