Applied Mathematical Sciences, Vol. 15, 2021, no. 7, 297 - 320
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2021.914488

Cryptocurrencies Markets and Entropy: A Statistical Ensemble Based Approach

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Abstract

In this paper, we consider the cryptocurrencies market and in particular, we try to point out an "affinity" between the system of agents trading in cryptocurrencies and statistical mechanics. We focus our study on the concept of entropy in the sense of Boltzmann and we try to extend such a definition to a model in which the particles are replaced by N economic subjects (agents). The agents are completely described by their ability to buy and to sell a certain quantity of cryptocurrencies. We provide some numerical examples by applying the model to the closing prices of the main six cryptocurrencies and we show that entropy can provide information about the position of economic agents in the phase space through the closing price.

Mathematics Subject Classification: 91G80, 28D20, 62P20, 82B30

Keywords: Cryptocurrency, Entropy, Boltzmann, Blockchain

1 Introduction

Trying to model the price trend of financial instruments has always been the focus of finance. Traditional theories are based on the search for significant variables that can explain the trend, such as Fama and French [13] that use expected returns to generate temporary price components; or Campbell and Shiller [6] that use accounting earnings data or Perasan and Timmermann [26] who consider the power of various macroeconomic factors on the markets. The evolution of the theory has pushed towards non-linear models such as Laffont, Ossard and Vuong [21] where an approach based on the method of nonlinear least squares is used or Qi [29] who proposes the use of neural networks based on linear regressions. Subsequently, when Brissaud [5] assimilated **entropy** to disorder, this instrument which had always been applied in physics also became part of finance.

Since the mid-19th century, entropy has been a key element linking mechanics to thermodynamics. The first to introduce this concept was Clausius [9], whose definition was applied to a thermodynamic system that performs a transformation; however, this entropy suffered from a conceptual problem which, as demonstrated by Gibbs [14], was revealed in the case of identical gases (*Gibbs Paradox*). He solved this problem by changing the count of states: in his definition, the number of states is based on the probability that each of these occurs during the fluctuations of the system. On the other hand, Boltzmann [4] presented his statistical interpretation of thermodynamic entropy, managing to link the macroscopic properties of a system with the microscopic ones. Based on Gibbs, in 1949 Shannon [33] developed a theory capable of evaluating the amount of information that is lost in receiving a message from a source to a recipient or, conversely, understanding what is the maximum possible compression of information without loss. This form of entropy was generalized by Rényi [30] and Tsallis [37] in a parameter-dependent Shannon entropy; applied by Adler [1] in the topology of dynamic systems as a measure of complexity; redefined by Pincus [28] (*approximate entropy*) usable for quantifying the regularity of information without knowing the system and - more recently - used by Chen [8] as a time series regularity measure.

It was the development of entropy in information theory that made it widely used in finance, in particular the generalizations of Shannon, Réyni and Tsallis contributed to creating a new line of application for the management of financial portfolios: Philippatos and Wilson [27] showed that entropy is a more effective indicator of standard deviation; Usta and Kantar [38] used an entropy model based on mean-variance for a better selection of the assets in the portfolio, to which Jana [18] added an objective function to generate a better diversification of the assets; Gulko [15] introduced the *Entropy Pricing Theory*, an alternative method for constructing risk-neutral probability measures without resorting to stochastic calculus; Nawrocki [24] defined a risk measure based on “weighted” entropy; Dionisio [11] demonstrated the greater validity of entropy on variance based on stock prices in the Portuguese market and Kirchner [19] argued that this represents a more effective tool for capturing

risk reduction through portfolio diversification. Ormos [25] on the other hand, combined entropy with CAPM's β to obtain a new risk measure and Sheraz et al. [34] used it to assess the volatility of the market indices.
In this paper, we want to demonstrate that it is possible to assimilate the system of cryptocurrencies to thermodynamic systems to be able to determine their entropy in the sense of Boltzmann so that we can make price predictions related to the possibility that they move in a more or less wide range; unlike all the recent applications concerning theories based on Shannon entropy and its derivations.
Innovation is linked to the reinterpretation of the monetary system of cryptocurrencies. In this sense, we can apply physical theories to a social science. It is interesting to develop this approach as we assume that the physical system described by adapting the economy can be summarized by the movements that cryptocurrencies perform the currency markets. Once the system has been described, our goal is to verify that entropy calculated in the physical sense also occurs in the economic context to allow us to make assumptions on how the process could move in the next future. Furthermore, the dataset we have considered support this hypothesis, therefore it seems reasonable to use this form of entropy to make assumptions about the future trend of prices. This type of conjecture has been presented by Sergeev [32] who proposes an unconventional representation of socio-economic balance, supplemented by Zakiras [40] which uses Newton's law of cooling. Finally, Khrennikov [20] analyzed the financial markets from a thermodynamic point of view (defining this approach as financial thermodynamics) and describing a financial *Carnot cycle*; Smith and Foley [35] have shown that a kind of parallelism exists between utility theory and thermodynamics and McCauley [22], based on this previous theory, maintains that the illiquidity of the markets does not allow for the application of the concepts of statistical mechanics.
The paper structure is the following: in Section 2 we analyze cryptocurrencies and their key characteristics, focusing on the fact that they have a supply limit; in Section 3 we describe the evolution of a system of a particle in statistical thermodynamics and how to determine its entropy, subsequently applying these notions to our monetary system; in Section 4 we define the theoretical assumptions we can link to the system created previously to study the price evolution in these currency markets and we analytically describe the calculation of entropy using real data; finally in Section 5 some conclusions are drawn.

2 Cryptocurrency

Cryptocurrencies represent a digital currency system with no guarantee institution and no transaction control. The main cryptocurrencies, by media

coverage or by the possibility that some financial intermediaries offer to use them as a payment instrument, are: Bitcoin, Ethereum, Ripple, Tether, Bitcoin Cash and Litecoin.
Unlike traditional financial assets, their value is not based on tangible assets such as the economy of a country or a company, but it is based on the security of an algorithm that tracks transactions. Their definition is controversial since by some entities [17] they are considered intangible assets (IFRS) while according to the German financial supervisory authority (BaFin) they are officially financial instruments [3]. Just as specified by Corbet et al. [10] the literature is still immature and new empirical and theoretical evidence continues to emerge monthly. Moreover, the same authors claim that in cryptocurrencies there are unique and specific issues that cannot be addressed using quantitative research and data mining as regulatory disorientation, cyber-criminality, and environmental sustainability.
All the cryptocurrencies have been based on the *Bitcoin*, a currency created by Nakamoto [23] who in 2009 released a software capable of implementing transactions. The currency itself is a unique alphanumeric string that represents a certain transaction, a transaction which will then be entered in a public register called *blockchain*. The transfer of the currency takes place through a digital signature mechanism by using the value of a function (called *hash function*) which is inserted in the previous transaction and guarantees its authenticity. The blockchain is the fulcrum of these systems and is essentially a register in which the data of the owners of the currency are entered, transactions occur in an encrypted manner. The blockchain is a data structure consisting of a list of transaction blocks linked together so that each refers to the previous one in the chain. Each block in the blockchain is identified by a hash generated using the SHA256 cryptographic algorithm on the block header. A block is a data structure that aggregates transactions to include them in the public register. The block is made of a header, containing metadata, followed by a long list of transactions. A complete block, with all transactions, is, thus, 1000 times larger than the block header [2]. The first identifier of a block is the cryptographic hash generated by the SHA256 algorithm, which returns, as a result, a 32-bit hash called *block hash*; the second identifier is the position in the blockchain called *block height*.
The cryptocurrency generation process is called *mining*, which adds money to the supply. Cryptocurrencies are "minted" during the creation of each block at a fixed and decreasing rate [2]: each block generated on average every 10 minutes contains new currency. For example, if we consider Bitcoin, every 210000 blocks the currency issue rate decreases by 50% (the availability of new coins grows as a geometric series every 4 years). It is estimated that around the year 2140, the production of the last block will be reached (6930000) and the number of coins produced will tend to its upper limit of 21 million (precisely

20999999.97690000), value introduced by Nakamoto himself and contained in the variable "MAX_MONEY" as can be read in the source code present on GitHub.[1] This value represents a sanity check, especially used to avoid bugs in which it is possible to generate currency from nothing. The integrity of the blockchain network is guaranteed through consensus algorithms such as *Proof-of-Work* (PoW) and *Proof-of-Stake* (PoS), that solve the Byzantine Generals Problem [7] (problem of consent in the presence of errors). A consensus algorithm is a mechanism used by the network to reach consensus, i.e. ensuring that the protocol rules are followed and that transactions occur correctly so that coins can only be spent once. The PoW (used in the Bitcoin system) starting from a node allows you to select the next one to be connected in the blockchain (through the hash link) based on a very complex computational algorithm that consumes resources especially in terms of energy of the machines to which it relies on the network; on the other hand, in PoS (used in the Ethereum system) the nodes are organized in the form of a stack in which they are linked to each other starting from the leader based on an election that involves various factors such as the age or status of a node. PoW is one of the most resistant (even if more expensive) algorithms, which linking a large number of blocks together make the hacking process too expensive.

3 Theoretical framework

The main assumption in this paper is that the prices of cryptocurrencies behave like a thermodynamic system, so it is possible to determine entropy by using the Boltzmann formula. In order to present the theoretical framework and the methodology, we need to briefly introduce the main physical results. In Statistical Mechanics a macroscopic system is made up of N molecules ($N \sim 10^{24}$ is the Avogadro's constant) whose mechanics provide the evolution of $6N$ dynamic variables describing completely the microscopic states of this system. Motion in the phase space can be studied using the $3N$ position components and the $3N$ momenta components, indicated with $\{\mathbf{q}_i\}$ and $\{\mathbf{p}_i\}$ whose evolution is driven by Hamilton's equations. We can use a compact notation to indicate a complete microscopic system of N particles:

$$\mathbf{X} \equiv (\mathbf{q}_1, ..., \mathbf{q}_N, \mathbf{p}_1, ..., \mathbf{p}_N) \tag{1}$$

where X is a vector of $6N$ real components. Mechanics, therefore, provides a very detailed description of the system contrary to thermodynamics which studies the collective variations; for this reason, the mechanical point of view can be defined *microscopic* and the thermodynamic one *macroscopic*. The study of the system from a microscopic point of view concerns experimental

[1]Source: https://github.com/bitcoin/bitcoin/blob/master/src/amount.h

observation on one or a few molecules.
Everything that happens from the microscopic side can be expressed in macroscopic terms through thermodynamics, defined in this case as a large amount of microscopic variables. We consider an isolated system of N particles described by the $3N$ coordinates and the $3N$ momenta in a $6N$-dimensional space at a certain time t. Particles are subject to the laws of classical mechanics and therefore $\mathbf{X}(t)$ evolves according to Hamilton's equations. Since the Hamiltonian $H(p, q)$ does not depend on time, the energy E is a conserved quantity during motion and develops on a fixed hypersurface. We want, for example, to measure an observable $A(\mathbf{X})$ (a function defined in the phase space) of the system in thermodynamic equilibrium, but since the scale of macroscopic times is much larger than the microscopic one, we can consider a datum as the result of a system that has gone through a large series of microscopic states; this implies that the observable must be compared with an average performed along with the evolution of the system calculated over very long times $\bar{A}$. The calculation of $\bar{A}$ would require knowledge of both the microscopic state at a certain moment and the determination of the corresponding trajectory in the phase space, which corresponds to a practically inexhaustible request. To determine the observable, the *ergodic* theory intervenes, according to which each energy surface is completely accessible to any motion with the given energy and the average residence time in a certain region is proportional to its volume. If these conditions are satisfied, the average $\bar{A}$ can be calculated as the average of $A(\mathbf{X})$ in which the states with the fixed energy contribute with equal weight. In applications it is convenient to consider on average all states with energy within a fixed range $[E, E + \Delta E]$; furthermore, we are only interested in some macroscopic properties such as particle number N and the volume V. There is an infinite number of systems that satisfy these conditions: these form the *Gibb's ensemble* which is represented by a set of points in the phase space characterized by a density function $\rho(p, q, t)$ defined so that $\rho(p, q, t)\, d^{3N}p\, d^{3N}q$ corresponds to the number of representative points of the system during the instant t contained in the infinitesimal volume of the phase space $d^{3N}p\, d^{3N}q$. Furthermore, since energy, volume and number of particles are constants of motion, the total number of systems in an ensemble is conservative.
We can thus introduce the *postulate of equal a priori probability* [12, 16] who claims that when a macroscopic system is in thermodynamic equilibrium its state can be with equal probability each of those which satisfies the macroscopic conditions of the system. This postulate implies that the system under consideration belongs to an ensemble called *microcanonic* with density function

$$\rho(p, q) = \begin{cases} \rho^* & if\ E < H(p, q) < E + \Delta \\ 0 & otherwise \end{cases} \tag{2}$$

where ρ^* is constant and all members of the ensemble have the same number of particles and equal volume.
We can define $\Gamma(E)$ the volume occupied by the microcanonical ensemble in the phase space as:

$$\Gamma(E) \equiv \int_{E<H(p,q)<E+\Delta E} d^{3N}p\, d^{3N}q \tag{3}$$

and $\Sigma(E)$ the volume bounded by the energy surface E:

$$\Sigma(E) \equiv \int_{H(p,q)<E} d^{3N}p\, d^{3N}q \tag{4}$$

so that

$$\Gamma(E) = \Sigma(E+\Delta E) - \Sigma(E). \tag{5}$$

Entropy, then, can be defined as:

$$\begin{aligned} S_\Gamma &= \int_{E\leq H\leq E+\Delta E} d^{3N}p\, d^{3N}q\, \rho(-\kappa_B \ln \rho) \\ &= \int_{E\leq H\leq E+\Delta E} d^{3N}p\, d^{3N}q\, \frac{1}{\Gamma}\left(-\kappa_B \ln \frac{1}{\Gamma}\right) \\ &= \frac{1}{\Gamma}\kappa_B \ln \Gamma \int_{E\leq H\leq E+\Delta E} d^{3N}p\, d^{3N}q \\ &= \frac{1}{\Gamma}\kappa_B \ln \Gamma \cdot \Gamma = \kappa_B \ln \Gamma(E) \end{aligned} \tag{6}$$

where $\kappa_B \sim 1.3806 * 10^{-23}$ is the Boltzmann constant. To analytically calculate $\Gamma(E)$, which represents the number of states accessible to the system at temperature T, we must consider that a microcanonical ensemble is made up of J identical copies of the closed system, each of which is located in a microstate $(\mathbf{p}_i, \mathbf{q}_i)$ of the phase space. Being all on the same hypersurface E, we can divide it into cells of equal size, where in each there are j_i systems such that $J = \sum_i j_i$. To define the system it is necessary to find the most probable distribution of the j_i microstates, that is, to count the total number of ways in which we can obtain a certain macrostate. In the Boltzmann paradigm with an ideal gas consisting of identical particles under the same conditions, we can say that

$$\Gamma(E) = \frac{J!}{\prod_i j_i!} \tag{7}$$

The idea that entropy is connected to volumes in the phase space finds its origin in the *Helmholtz Theorem*, whose goal is to exactly bring thermodynamics down from mechanics. Considering a one-dimensional mechanical system with Hamiltonian:

$$H(p,q,\tilde{V}) = \frac{p^2}{2m} + \phi(q,\tilde{V}) \tag{8}$$

the Helmholtz theorem defines

$$\tilde{S}(E,\tilde{V}) = \kappa_B \ln \oint p(q)\, dq \tag{9}$$

$\tilde{S}(E,\tilde{V})$ is proportional to the logarithm of the phase space area enclosed by the energy orbit E and parameter $\tilde{V}$; it can, therefore, be written in the form

$$\tilde{S}(E,\tilde{V}) = \kappa_B \ln \int_{H(p,q,\tilde{V})<E} dp\, dq \tag{10}$$

To generalize the result to systems with N particles Boltzmann makes the ergodic hypothesis. Assuming this hypothesis it is possible to show the generalized Helmholtz theorem:

$$\tilde{S}(E,\tilde{V}) = \kappa_b \ln \int_{H(\mathbf{p},\mathbf{q},\tilde{V})<E} d\mathbf{p}\, d\mathbf{q} \tag{11}$$

Let us now try to translate this physical theory into a financial dress.

4 The model

A first attempt to create a link between economics and thermodynamics was that of Saslow [31], who based the relationship on the utility function U (assumed in complete analogy with thermodynamics) and verified how different types of relationships at the base of the economy were in accordance with the assumptions of thermodynamics (e.g. the number of transactions is conservative, as in the case of the total energy of a system). On this basis, Viaggiu et al. [39] have developed a representation of an economic model relating to money from a thermodynamic point of view. In their description the ensemble is made up of the N interacting economic subjects, entirely described by two variables $\{x_i, y_i\}$ which represent money and credit/debt capacity and which are not conjugated in the sense of mechanics Hamiltonian. The key characteristic is to consider a representative function of the total currency as a conservative law, to be able to exploit the ergodic hypothesis.
Our idea is to go back to their hypothesis by applying it to the case of cryptocurrencies and the goal is to create an *agent-based* model, in which the particles are replaced by N economic subjects (agents) who intend to trade in cryptocurrencies (compared only to a reference currency, such as the USD) where basically it is possible to determine the movement of economic subjects in a certain "phase space" and whose entropy provides a **proxy** for this movement. We can also fully describe an economic agent in our phase space by 2 variables, which we can, however, identify as $\{x_i, y_i\}$, where x_i and y_i indicate,

respectively, the ability to buy and to sell a certain quantity of cryptocurrencies (both expressed in monetary terms). In this phase space (following Saslow's theory [31]) the value that satisfies the utility function is the number of Bitcoins purchased (by definition the number of these cryptocurrencies in circulation is fixed and are purchasable only portions of this number). To better explain how we defined the 2 key variables, we can give an example. Suppose an economic agent wants to buy a certain number of Bitcoins, e.g. 500 and has di 1000000 USD in his bank account. To buy them in the first place he will need to find someone willing to sell such a quantity of Bitcoin (by definition the market for these cryptocurrencies needs direct exchange between those who already own a certain quantity and those who want to buy) and he will find a certain price that e.g. will allow him to buy only 300 of the 500 Bitcoin he needs (in this sense the economic agent is subject to the price). His ability to buy is therefore understood as the ability to buy a number of cryptocurrencies to reach a certain utility (which in this case has not been completely satisfied). This also happens because, according to the physical theory of Boltzmann's model, the economic agent (particle in the physical case) must "collide" with another agent (particle) that is in its own microstate. The latter hypothesis is possible according to the fact that the market to which we refer is influenced only by the supply and demand leverage, and that generally those who operate in these markets negotiate almost exclusively in relation to the number (portion) of cryptocurrencies and not for the price. As for [39], even if the complete Hamiltonian formalism is not respected, we can consider as a conserved quantity the total number of cryptocurrencies in circulation which by their definition is constant over a suitable time interval through the function $M(x_i, y_i)$ (as in the particular case of Bitcoins for which the supply limit is fixed at 21 million). However, since the supply limit has not yet been reached by any cryptocurrency we consider this quantity constant concerning the currency in circulation in a precise time t, therefore:

$$M = \sum_{i=1}^{N} x_i + y_i. \tag{12}$$

In this sense, the sum of the ability to sell and buy of the N agents fully describes the cryptocurrencies in circulation. The ergodic hypothesis allows us, given a certain function $f(x_i, y_i)$, to express its average with respect to the time in terms of an average over the ensemble at fixed M:

$$\bar{f} = \int_{M=const} f(x,y)\rho(x,y)\,dx\,dy \tag{13}$$

where $\rho(x, y)$ denotes the probability distribution of the ensemble. Through these assumptions we can verify the economic transformations through thermodynamics; in particular, as in statistical mechanics, we can calculate the *volume in the phase space* [39]. If we integrate over all the available volume of the configuration space spanned by $\{x, y\}$ with $\bar{M} = m$ (where $\bar{M}$ denotes the average over the whole configuration space) we have $\int_{\bar{M}=m} d^N x \, d^N y = 0$. So introducing a thick shell Δ where $\Delta \ll m$ we can define:

$$\Gamma(m) = \int_{m<M<m+\Delta M} \frac{d^N x \, d^N y}{k^{2N}} \tag{14}$$

where $d^N x \, d^N y$ is understood as the phase space and k is a normalization factor such that Γ is dimensionless. This functional represents the number of microscopic realizations of the system under examination and allows us to calculate the entropy S as described in the equation (6).

We are therefore interested in verifying how economic subjects move in the phase space we have created (in relation to their propensity to buy or sell) and in order to exploit entropy as a proxy we can use price dynamics as an indicator (obtainable from the currency markets, FOREX).

First, we know that cryptocurrencies are used by an approximate number of economic entities equal to 44 million (Szmigiera [36] estimates that the number of blockchain-based portfolios is 44 million so we can assume that there is an at least equal share of subjects who intend to trade between all the different types of existing cryptocurrencies) for which $N \gg 1$. We also know that every subject in our system is fully described by its ability to buy and sell ($\{x_i, y_i\}$). Let us consider that these two variables are **summarized in the last prices** of the cryptocurrency on the currency markets, a type of price used to keep track of changes in the value of an asset throughout a session. In this sense, the latest prices allow us to understand whether, compared to the previous session, the ability to buy or sell prevailed. We can summarize this price capability in the sentence *"prices describe the strength with which agents position themselves in the phase space"*. We have not identified a function such that a change of x_i and y_i leads to a change in price, however the economic subjects move in relation to the quantity purchased/sold. The price in this case is a summary value of the movement of the set of economic entities: suppose e.g. that on day 1 price is 1000 USD of a portion of Bitcoin. On day 2 we observe that the price has risen to 1100 USD and we can assume that this increase is due to a greater quantity of subjects who want to buy cryptocurrencies (in terms of the variables x_i and y_i, we can say that the capacity to buy subjects has increased, increasing x_i). On the 3 day, however, we note that the price of Bitcoins is 900 and what we can assume is that the reduction is due to a greater number of subjects who want to sell (increasing y_i). The key feature is that we do not make inferences about price dynamics, but about the movement of agents in the phase space through price (as an indicator). Moreover, we can use the function M (described above) because in a certain time t the quantity of cryptocurrencies is constant and quantifiable, in this way we can go back to the previous economic model and determine Γ as described in the equation (14). Analytically, we do not consider the number of economic subjects present in the market but indirectly deduce their "position" in the phase space from the difference between the closing prices. The process that led to the definition of the results is the following:

- We take a certain reference interval (5 days) and cluster the closing price series based on this interval;
- For each cluster there is a maximum and a minimum price, we calculate the difference in terms of necessary steps to pass from one to the other obtaining a certain value of **gap G** (this assumption is based on the idea that the distance between maximum and minimum is a measure of the dispersion of agents in our phase space);
- We use combinatorial analysis considering the value used for clustering to determine the "volume" occupied by the disposition of the agents, therefore:

  $$\Gamma = G^5 \tag{15}$$

  This equation derives from equation (7) in Boltzmann's formulation to determine the number of microstates; however, since we only consider the arrangement of the economic subjects in the phase space and we do not have the problem of the indistinguishability of the particles (as in the physical case), our number of microstates is defined simply as the combination of the different time steps considered between the price min and max of the cluster.

Once the value of Γ is determined, entropy can be calculated by using the Boltzmann formula:

$$S = \kappa_B \ln \Gamma. \tag{16}$$

As for [31], the use of the Boltzmann constant is not a necessary condition since in a certain region we will never have a number of economic subjects comparable to an Avogadro number of molecules, however its use we believe may be an element help in improving the entropy determined in our way. Finally, we can *"rationalize"* this entropy value obtained by multiplying it by 10^{23} to make the value more readable also from a graphic point of view (e. g. to get 46.6 instead of 0.0496).

The entropy determined in this way is an additive type measure which at the base provides for the presence of a logarithm, as it transforms a multiplicative type phenomenon into an additive one. In literature there are functions that seem similar, but have completely different purposes: our entropy is not a measure of price volatility and in this sense the argument of the logarithm is completely different from those generally used in economic sectors; in this case with a logarithmic function we use an additive function starting from a multiplicative concept (like microstates) since we have the same requirement as Boltzmann in considering microstates. On the basis of the numerical analysis carried out we can say that, since entropy is always growing in accordance with the second principle of thermodynamics, if in one part of the system we observe a reduction necessarily in another part entropy must grow (a classic example concerns tidying up a room: in this case, the entropy of the room is reduced but I have increased the overall one because I used energy to tidy up, increasing the entropy of those who are tidying up). In agent-bases logic the number of transactions could be used (instead of the price), but probably it would not add any additional information to our model.

4.1 Dataset

The empirical analysis has been applied to the closing prices of the main six cryptocurrencies[2], all related to the US dollar (USD), that are:

- Bitcoin, whose price with 1 decimal digit provides for a tick size equal to 0.1;
- Ethereum, whose price with 2 decimal places provides for a tick size equal to 0.01;
- Ripple, whose price with 5 decimal places provides a tick size equal to 0.00001;
- Tether, whose price with 4 decimal places requires a tick size equal to 0.0001;
- Bitcoin Cash, whose price with 2 decimal places requires a tick size equal to 0.01;
- Litecoin, whose price with 3 decimal places requires a tick size equal to 0.001.

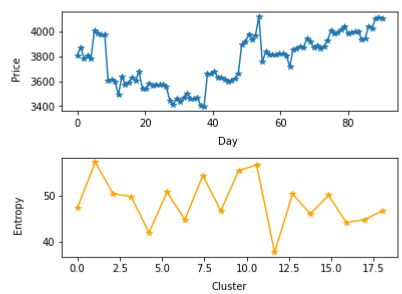
Prices are considered with a daily time frame over 1 year, from 1/1/2019 to 31/12/2019 and they are clustered in 5 days. To make the figures more clear, the 1-year interval has been divided into 4 quarters. Furthermore, to better test the idea, the same test was carried out also on daily prices at 1 minute of 1/4/2020 recorded from 10:56 to 11:52, instead of clustered in 5 minutes. The difference from the daily case is that these prices were collected, always from the same source, but observed on different currency markets; in particular Bitcoin on the GDAX exchange, Ethereum on Bibox exchange,

[2]Source: Investing.com

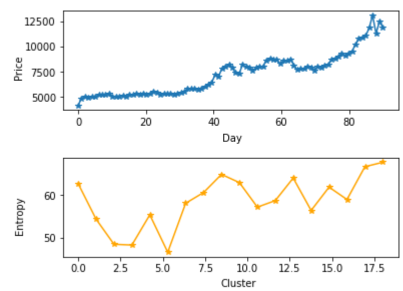
Ripple on Binance exchange, Tether on Kraken exchange, Bitcoin Cash on Huobi exchange and Litecon on ZB.COM exchange.

4.2 Numerical results

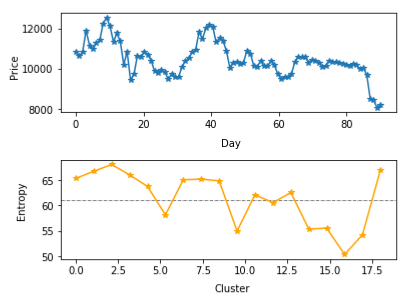
We can start the analysis from the annual case. The first cryptocurrency analyzed is Bitcoin (BTC/USD). We distinguish the trend of entropy compared to prices in the 4 ranges previously defined:



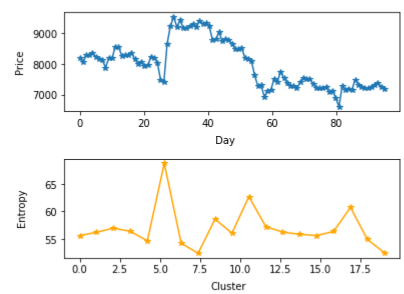
(a) Closing prices and entropy 1/1 - 31/3



(b) Closing prices and entropy 1/4 - 29/6



(c) Closing prices and entropy 30/6 - 27/9

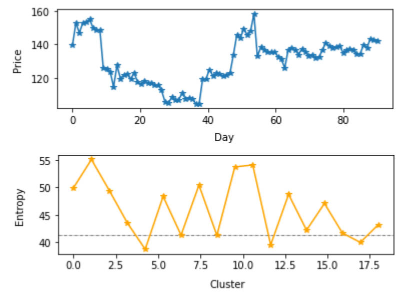


(d) Closing prices and entropy 28/9 - 31/12

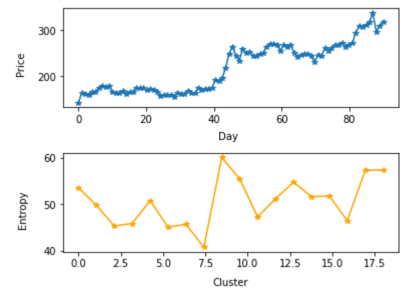
Figure 1: Prices (blue) and entropy (orange) Bitcoin in the period 1/1 - 31/12

As can be seen graphically, when entropy reaches a point of relative minimum falling below a certain threshold (it therefore undergoes a sharp reduction) it is forced in the next cluster to grow, almost as if to rebalance itself. This hypothesis does not seem to occur in the case of entropy lowering only a few points, as seen in the figure 1(c) in the case of clusters 3, 4 and 5. This implies that in the cluster in which the entropy descent occurred there was a very small gap and at a decrease in entropy in prices necessarily another part of the system is simultaneously experiencing an increase in entropy, which will cause a short-term effect. The last result, translated into our model created previously, indicates that in clusters in which entropy drops drastically, economic subjects concentrate in a relatively small "volume".

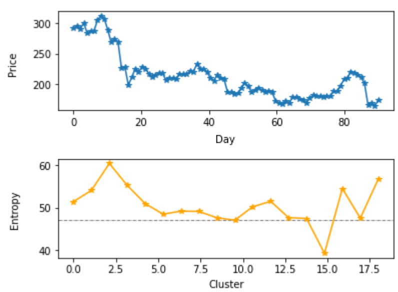
The second cryptocurrency analyzed are Ethereum (ETH/USD) as in the previous case:



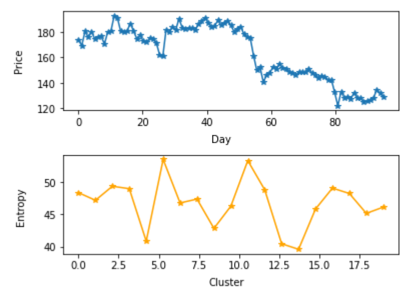
(a) Closing prices and entropy 1/1 - 31/3



(b) Closing prices and entropy 1/4 - 29/6



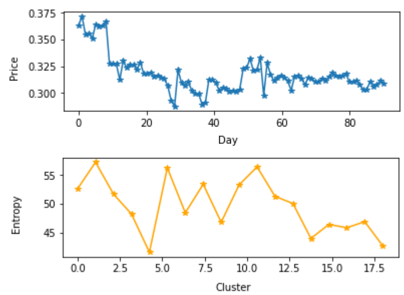
(c) Closing prices and entropy 30/6 - 27/9



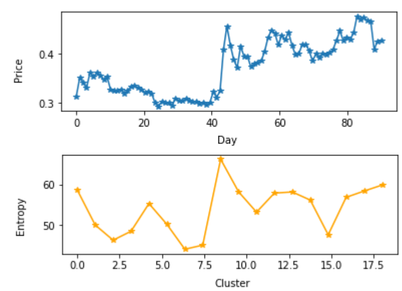
(d) Closing prices and entropy 28/9 - 31/12

Figure 2: Prices (blue) and entropy (orange) Ethereum in the period 1/1 - 31/12

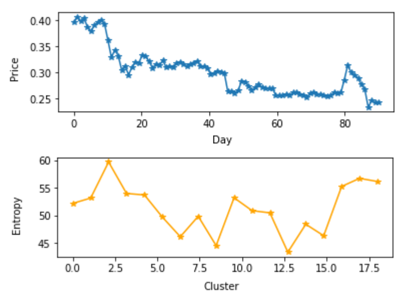
Again, especially as seen in the figure 2(c) when entropy decreased sharply after a period of standing (small ups and downs) in the following period it was "forced" to grow (e. g. as happens in the case of excitation of the particles contained in a gas). A particular situation, however, occurs in the figure 2(a) and in particular in clusters 2, 3 and 4: in this case entropy continued to fall despite having suffered a sudden movement. This situation allows us to highlight 2 things: that the sharp drop comes from a situation where the gap between the max and min prices is lower than a certain threshold value and that probably the descent of entropy should not be considered only in the passage from one cluster to another, but in the passage from groups of clusters; in fact, if we consider clusters 2 to 5 as a single group, it is easy to see how entropy has undergone a really sharp reduction followed by growth.
The third cryptocurrency is represented by Ripple (XRP/USD):



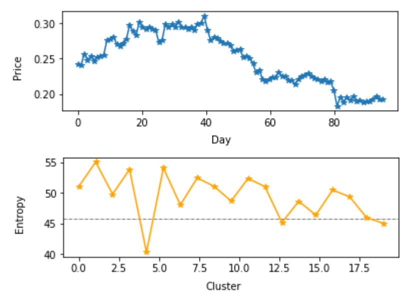
(a) Closing prices and entropy 1/1 - 31/3



(b) Closing prices and entropy 1/4 - 29/6



(c) Closing prices and entropy 30/6 - 27/9

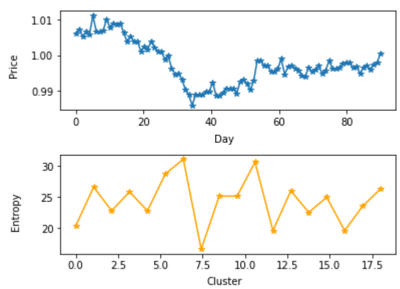


(d) Closing prices and entropy 28/9 - 31/12

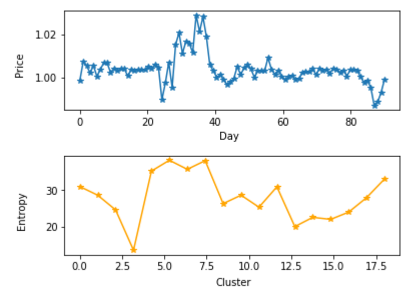
Figure 3: Prices (blue) and entropy (orange) Ripple in the period 1/1 - 31/12

The situation is not very different from the Ethereum, but the marked variations are related to the fact that this cryptocurrency moves in a price range $[0, 1]$ for which every movement is important. As seen in the figure 3(d) in clusters 4, 5 and 6 the entropy has returned to its original level following a sharp fall. We expected a growth but the fact that it has grown so much is related to the range in which prices move (as if it were a smaller volume than in previous cases).

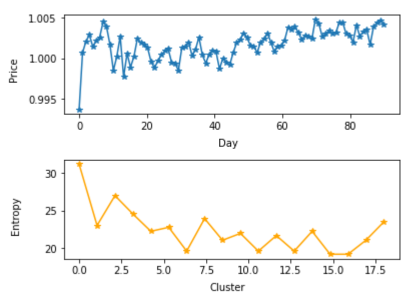
The fourth cryptocurrency analyzed is the Tether (USDT/USD), whose price moves in a neighborhood of 1 and consists of 4 decimal places:



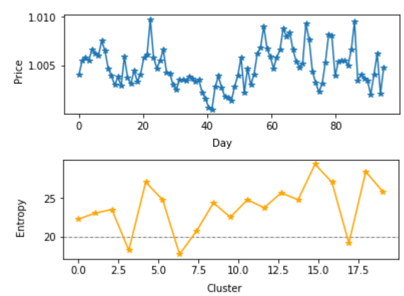
(a) Closing prices and entropy 1/1 - 31/3



(b) Closing prices and entropy 1/4 - 29/6



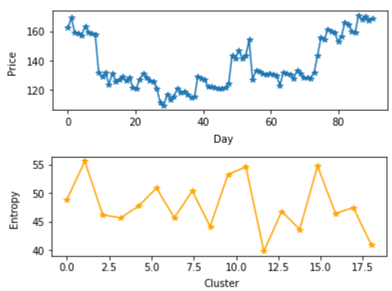
(c) Closing prices and entropy 30/6 - 27/9



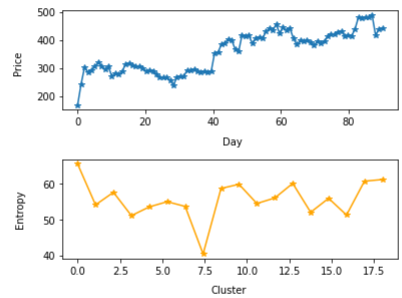
(d) Closing prices and entropy 28/9 - 31/12

Figure 4: Prices (blue) and entropy (orange) Tether in the period 1/1 - 31/12

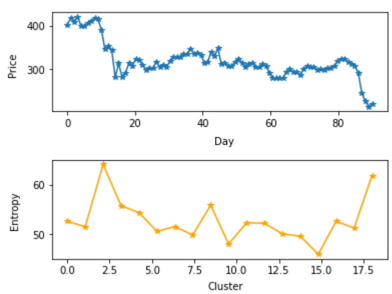
As in the previous case, the range of variation of prices is very "narrow" and every movement is important. In this case, however, it is possible to notice for example looking at the figure 4(d) what is the gap value and therefore the entropy threshold that, if "under"-passed, will cause an immediate growth in the next future.
The fifth currency analyzed is Bitcoin Cash (BCH / USD):



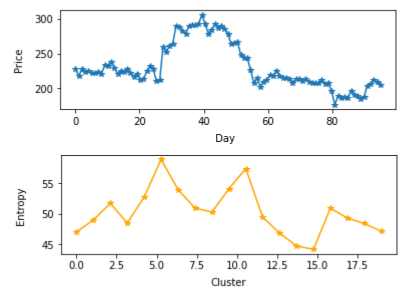
(a) Closing prices and entropy 1/1 - 31/3



(b) Closing prices and entropy 1/4 - 29/6



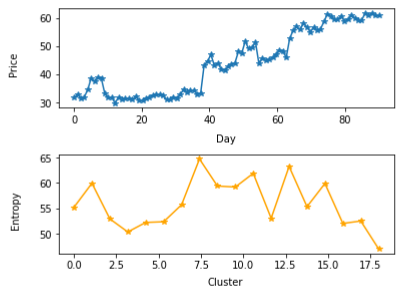
(c) Closing prices and entropy 30/6 - 27/9



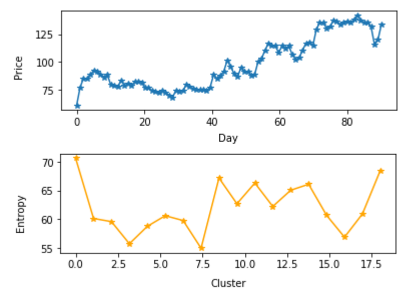
(d) Closing prices and entropy 28/9 - 31/12

Figure 5: Prices (blue) and entropy (orange) Bitcoin Cash in the period 1/1 - 31/12

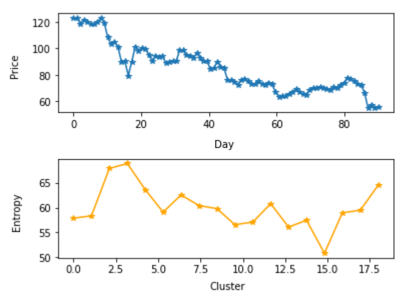
In this case the figure 5(d) shows how the gap threshold below which a sharp drop in entropy occurs can also be quite high (especially in currencies where high volatility allows it to to move many points from one price to another). This situation, in analogy with physics, occurs for example in the case of gases that need a much stronger heat source than other types.
The last cryptocurrency we have considered is Litecoin (LTC/USD):



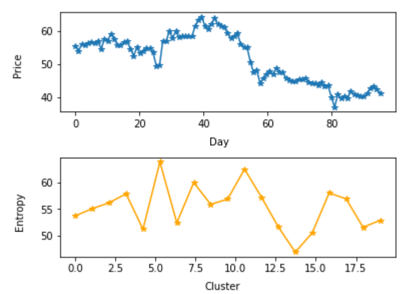
(a) Closing prices and entropy 1/1 - 31/3



(b) Closing prices and entropy 1/4 - 29/6



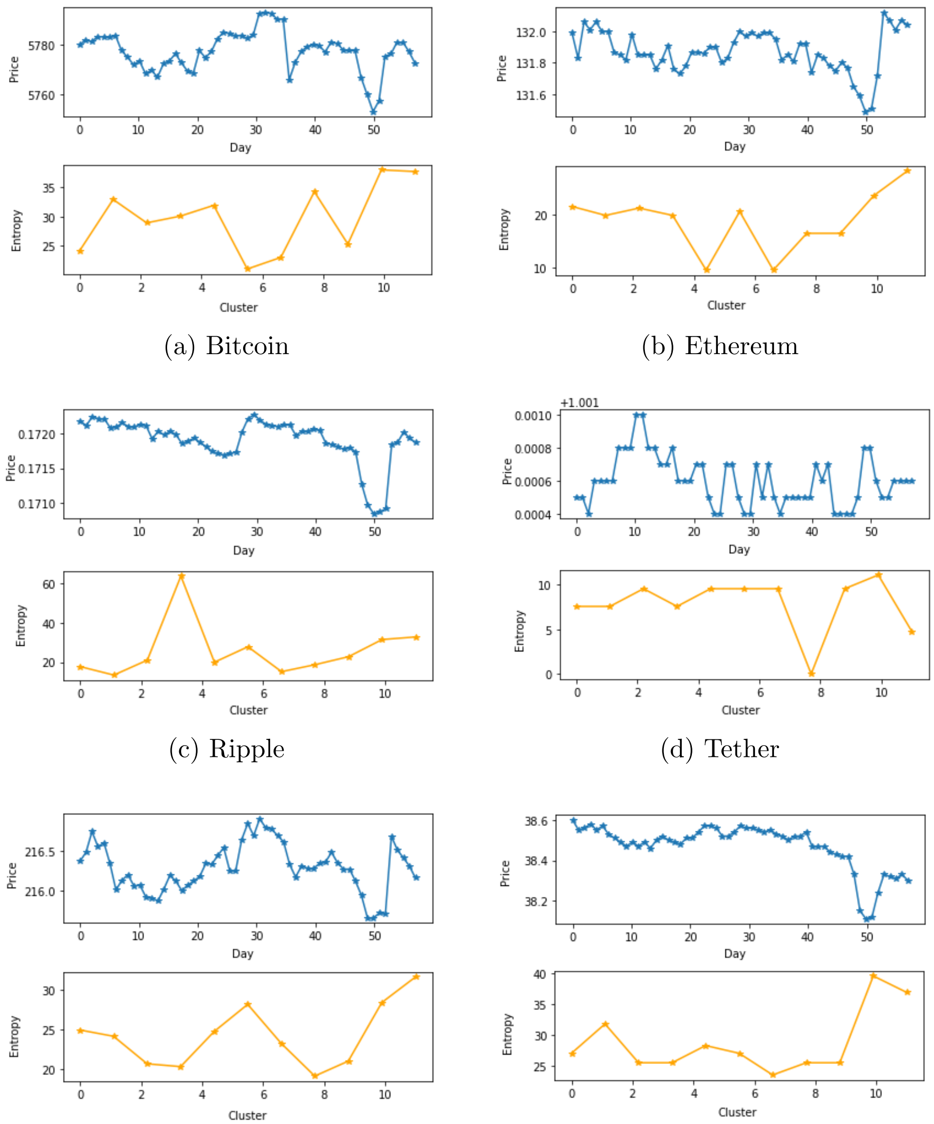
(c) Closing prices and entropy 30/6 - 27/9



(d) Closing prices and entropy 28/9 - 31/12

Figure 6: Prices (blue) and entropy (orange) Litecoin in the period 1/1 - 31/12

Also in this cryptocurrency all the situations defined above occur, in particular from the figure 6(d) it can be seen how, following the fact that the first 4 clusters are growing despite the gap value being quite low, the gap threshold to define the drastic descent of entropy is quite low.
As for the case of 1-minute prices, we can summarize the trend of the different cryptocurrencies together as shown in figure 7 which shows how all the assumptions made in the previous case are also respected for prices of this type.



(a) Bitcoin (b) Ethereum

(c) Ripple (d) Tether

(e) Bitcoin Cash (f) Litecoin

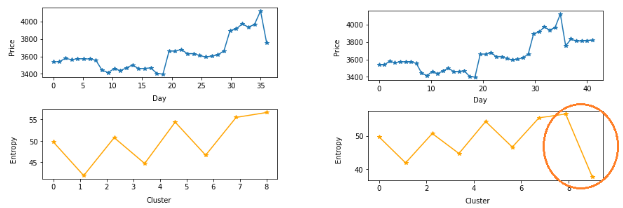
Figure 7: Prices (blue) and entropy (orange) of cryptocurrencies based on 1 minute

All these evidences confirm the relationship between the phase space created previously and statistical mechanics. On this basis we can expect behavior of economic subjects very similar to particles, therefore in situations in which entropy is drastically reduced we expect subsequent growth situations since entropy is continuously increasing, with the consequence that economic agents from a situation of concentration (which occurs when entropy is reduced) they begin to disperse again in the phase space (as in the case of gases) and this movement is summarized by the price. For example, suppose we are in a cluster C where entropy has declined sharply. As previously defined, we expect entropy to grow in the next cluster and this leads to an increase in the price gap. The hypothesis we can make is that the value of the gap in the cluster $C+1$ is at least one unit higher than the value in the cluster C. Let's consider a series of clusters in which we know the trend of prices and entropy (in this case we have considered an extract of the last Bitcoin price in the period 1/1/2019 - 31/3/2019). We know that in cluster 5, entropy has greatly reduced so we expect it to grow in cluster 6, creating a greater gap between prices than the previous one. At this point, knowing the value of the gap in the cluster C, we can create a bifurcation that represents the possible evolution of the movement of agents based on price dynamics in the event of a bullish or bearish trend. Suppose a gap value 4 times larger than the previous one (as happens in reality) and a first cryptocurrency price close enough to the last price of the previous cluster; what we can expect is such a situation:

- If the second closing price of the cluster $C+1$ is **higher** than the previous price in the same cluster and assuming an upward trend we can assume that the series of prices continues in an area that we have defined as Gap^{-};
- If the second closing price of the cluster $C+1$ is **lower** than the previous price in the same cluster and assuming a bearish trend we can assume that the price series continues in an area that we have defined as Gap^{+}.

In any case deriving from the bifurcation, the gap (therefore the number of microstates) influences the possible dispersion of the economic agents in the phase space.

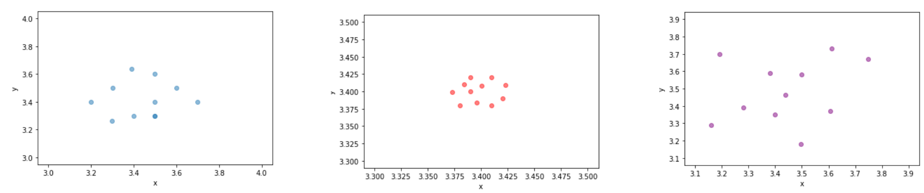
At this point, to make the idea better, we can represent the two key variables of the model through a scatterplot in which the points represent a simplification of the number of economic subjects and their position. Furthermore, we can compare how agents move in relation to the entropy variations highlighted by the respective gaps and therefore by the price (in particular prices and entropy represent a section of the first quarter of Bitcoins).



(a) Stationary price and entropy

(b) Descending entropy

(c) Growing entropy



(d) Economic agents position in stationary case

(e) Economic agents position in decreasing case

(f) Economic agents position in increasing case

Figure 8: Comparison between the position of economic agents and relative entropy

5 Conclusions

In this paper, we have shown how it is possible to apply Boltzmann's entropy to cryptocurrencies. We have defined a similarity between a thermodynamic system and a currency system based on cryptocurrencies characterized by the presence of N subjects interested in buying (or selling) this type of currency. Assuming that the quantity of money at a certain moment t is fixed and determinable, it is possible to hypothesize that the position of each economic entity is summarized by the last price of the cryptocurrency itself in the currency markets, as an indicator characterized by the ability to buy and sell. With this hypothesis, it was possible to determine the entropy using the Boltzmann formula, in particular, its calculation was made by dividing the time interval into clusters and calculating the gap between the different prices. This analysis has shown that when entropy falls sharply the economic subjects tend to approach each other in the phase space identified by us and, in accordance with the second principle of thermodynamics, they must necessarily move away, which leads entropy to necessarily grow. The next step is to try to model the stock price in this way so that we can apply the same type of entropy and look for a similar result.

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Received: May 3, 2021; Published: May 27, 2021