Forward and Reverse Parking in a Parking Lot

Kexin Xie

School of Mathematics and Statistics
Dalian University of Technology
Dalian 116024, Liaoning, China

Myron Hlynka

Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario, Canada N9B 3P4

Abstract

The choice of forward and reverse parking in a parking lot is studied as a stochastic process. An $M/M/c/c$ queueing system is used as an initial framework. We use Monte Carlo simulation to get the relationship between vehicle orientation and vehicle entry and exit rates, as well as the most likely parking states at each specific rate.

1 Introduction

There is considerable mathematical research on parking lots. See [2], [7], for example. However, there does not appear to be a mathematical study on reverse parking versus forward parking, in parking lots. There are two possible parking directions for a vehicle in a parking lot: forward parking and reverse parking (or backward parking). See Figure 1. Reverse parking is considered safer than forward parking. From the USA FIRE PROTECTION official website ([5]) "Reverse parking is about making the environment safer when the

Forward Parking   Reverse Parking
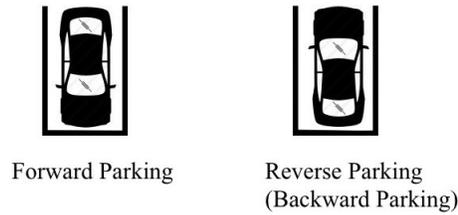   (Backward Parking)

Figure 1: Forward and Reverse Parking

driver leaves the parking space. When reverse parking, a driver is going into a known space with no vehicle and pedestrian traffic. When leaving the parking space the driver is able to see the surroundings more clearly". In [6], it is suggested that people should reverse park because reversing into a parking space makes it easier to maneuver out of the space. "If you think reverse parking is difficult, it's much more difficult to try and wiggle your car out of a tight space when you have parked front first."

The City of Mount Pearl, Newfoundland adopted a Reverse Parking Policy ([3]) It says " all employees parking vehicles on City of Mount Pearl property and on any other parking lot must ensure to park the vehicle safely so you can exit the parking lot by pulling out head-on." Some people are not willing to reverse park. "In a study conducted by The National Highway Traffic Safety Administration (NHTSA), on average 76% or drivers park nose-in, the way many of us are used to parking." ([5]) In a double-row parallel parking lot, people have a high probability of not parking backward when the opposite parking space is already occupied; in a single-row parking lot, people are less willing to reverse park because it would take more time, and slow down or vehicles directly behind.

In this article, we will build a mathematical model to simulate the parking orientation in the parking lot and discuss the relation between parking orientation and arrival time and service time. We use Markov Chain methods to determine the effectiveness of our model. This article appears to be the first mathematical model to study reverse parking in parking lots.

An important property of an exponential random variable X is the memoryless property. This property states that for all $x \geq 0$ and $t \geq 0, P(X > x + t | X > t) = P(X > x)$. This property is used in our simulation to select the next event time. We also use the well-known properties below ([4]) , in our simulation programming.

Theorem 1.1. *Let X,Y be independent random variables. If $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$, then*

$$W = \min(X, Y) \sim Exp(\lambda + \mu).$$

Theorem 1.2. *Let X,Y be independent exponentially distributed random variables with parameters λ and μ. Then $P(X < Y) = \frac{\lambda}{\lambda+\mu}$.*

2 Mathematical Model

In this section, we describe our model. Assume vehicles arrive according to a Poisson process, with exponentially distributed sojourn times. Each parking space is considered as a server. When all spaces are occupied, any arriving vehicle is lost to the system. This gives an $M/M/c/c$ queueing system.

2.1 Setting (Assumptions)

- The system we study consists of a simplified parking lot with pairs of parking spots, shown vertically below, and access lanes on both sides. In the general case, there are n parking spaces with m rows of 2 cars per row so $n = 2m$. We will work with a particular case with $n = 10$ parking spots in $m = 5$ rows. For example, matrix A represents a specific parking situation in the parking lot. The larger black section for each vehicle represents the front of the car. We use the notation 1 for parking forward, -1 for reverse parking, 0 for empty. See Figure 2.
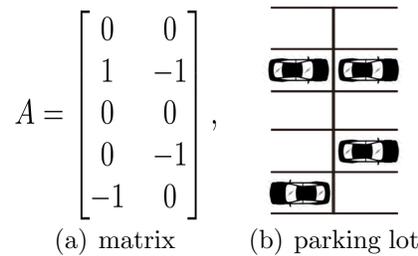
$$A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix},$$

(a) matrix   (b) parking lot

Figure 2:

- Vehicle arrivals occur at rate λ according to a Poisson process. Parking times are exponentially distributed with mean $1/\mu$. This parking lot can be viewed as a multi-server queueing system, with vehicles being customers and the parking spots acting as servers.

- The system runs for time T. In our model, we assume $T = 24$ hours.

- If there is no free parking space, an arriving vehicle leaves the parking lot and is lost to the system.

- There are two ways that a vehicle can achieve a reverse parking or backward parking position. One way is for the vehicle to back into an empty parking position. Another way is the for the vehicle to select an empty pair of spots and to drive through one spot to position itself in a "reverse" parking position in the other spot.

- We assign different probabilities for different parking spaces. When a parking space is occupied, the parking probability of an arrival is 0. When there is a parking spot with no car in the opposite pair position, we choose the parking probability to be twice as large as when there is a car opposite. (This ratio can be treated as a parameter.) For our example, the elements in matrix B represent the parking probability of each space in matrix A:

$$B = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0 \\ 0.2 & 0.2 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

- For a specific parking space, we assume the probability of parking forward is 0.8, backward is 0.2 when the opposite pair position is occupied. When the opposite pair position is empty, assume that the probability of parking forward is 0.1, backward is 0.9. These probabilities can be taken to be parameters of the model.

2.2 Markov Analysis

An M/M/c/c queueing system has a state space $\{0, 1, 2, 3...c\}$ where the state represents the number of vehicles in the system. The following is well known. ([8], [1]). See Figure 3.

" The state-space diagram for M/M/c/c is shown as below:

Figure 3: Process diagram for M/M/c/c Queue

The transition rate matrix is:

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -(\mu + \lambda) & \lambda \\ & 2\mu & -(2\mu + \lambda) & \lambda \\ & & 3\mu & -(3\mu + \lambda) & \lambda \\ & & & & \ddots \\ & & & & (c-1)\mu & -(\lambda + (c-1)\mu) & \lambda \\ & & & & & c\mu & -c\mu \end{bmatrix}$$

The steady-state distribution exists with probabilities below. See [8].

$$P_k = P_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} = P_0 \frac{(\rho c)^k}{k!}, k = 1, 2, \cdots, c$$

$$P_0 = \left(\sum_{k=0}^{c} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}\right)^{-1} = \left(\sum_{k=0}^{c} \frac{c^k}{k!}\rho^k\right)^{-1}, \text{ where } \rho = \frac{\lambda}{c\mu}.$$

In our parking lot model where $c = 10$, we illustrate the values of $P_k, k = 0, 1, 2, \cdots, 10$ if $\lambda = 1$, $\mu = 1$. Results are as shown in Table 1, and our simulated results give close approximations.

Table 1: Calculated and simulated values of P_i

P_i	P_0	P_1	P_2	P_3	P_4	P_5
Calculated	.368	.368	.184	.0613	.0153	.00307
Simulated	.368	.367	.184	.0617	.0153	.00311
P_i	P_6	P_7	P_8	P_9	P_{10}	
Calculated	.000511	.00007	.000009	0	0	
Simulated	.000492	.00008	.000008	0	0	

Up to this point, the analysis has not considered the orientation of the vehicle (forward or reverse).

2.3   Oriented Model

We have six different types of oriented pairs for each row of the parking lot matrix: Single Forward, Single Backward, One Forward and One Backward, Double Forward, Double Backward, or Empty, as the figure shows below (here written horizontally): See Figure 4.
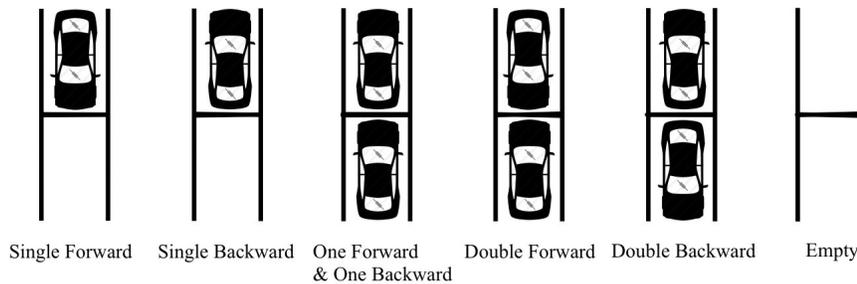

Single Forward   Single Backward   One Forward   Double Forward   Double Backward   Empty
                                   & One Backward

Figure 4: All possible pairs

Now we want to calculate the number of possible states for our oriented parking lot model with n spots where $n = 2m$ with m rows of 2 spots (pairs),

where each state is a six tuple with the six components giving the count of the six possible pair types. We are only concerned with the counts of each orientation type and not the order in which they occur.

Theorem 2.1. *The number of states (6 tuples) in an oriented parking lot with $n = 2m$ spots consisting of m pairs is $\binom{m+5}{5}$.*

Proof. Let $a_1, a_2, a_3, a_4, a_5, a_6$ represents the count of each pair type. The total number of pairs is m so

$$\begin{cases} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = m \\ a_i \geq 0, \quad i = 1, 2, \cdots, 6 \end{cases}$$

So the number of states is the number of integer solutions. We use the standard combinatorial method. ([4]):
In our example, with $n = 10$, we have $m = 5$. we represent the pairs by stars.

$$\star\ \star\ \star\ \star\ \star$$

We add 1 to each a_i and let $b_i = a_1 + 1$. Then

$$\begin{cases} b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = m + 6 \\ b_i \geq 1, \quad i = 1, 2, \cdots, 6 \end{cases}$$

Adding 1 to each of the six a_i results in $m + 6 = 5 + 6 = 11$ stars.

$$\star\ \star\ \star\ \star\ \star\ \star\ \star\ \star\ \star\ \star\ \star$$

To find the number of solutions for the b_i we have to insert $k - 1$ separating bars in the spaces between the stars to get k groups of at least one star each. In our model $k = 6$, since there are 6 possible orientations.

$$\star\ |\ \star\ \star\ |\ \star\ \star\ \star\ |\ \star\ |\ \star\ \star\ |\ \star\ \star$$

Thus the number of solutions for the equation $b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = m + 6$ with $b_i \geq 1$ is the number of choices for insertion of the $k - 1 = 6 - 1 = 5$ separating bars. So the number of solutions is $\binom{m+5}{5}$. This also gives the number of solutions of $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = m$ with $a_i \geq 0$, and the result follows.   □

We can see how quickly the number of states increases. See Table 2.
In our case, with $n = 10$ parking spaces in $m = 5$ rows, there are $\binom{10}{5} = 252$ possible oriented parking configurations. This is large so it makes sense to move to simulation. The transitions between states are somewhat complex and we illustrate by looking at a particular state $(1, 0, 0, 0, 0, 4)$, which represents 1 Single Forward, 0 Single Backward, 0 (One Forward and One Backward), 0 (Double Forward), 0 (Double Backward), and 4 Empty pairs. A departure of a

Table 2: Counting the number of possible states

n	m	$\# \, of \, states$
2	1	6
4	2	21
6	3	56
8	4	126
10	5	252

vehicle occurs with rate 1μ since there is only 1 vehicle in the parking lot and the new state will be $(0,0,0,0,0,5)$. Arrivals to the state $(1,0,0,0,0,4)$ will transition to one of the states $(2,0,0,0,0,3)$, $(0,0,0,1,0,4)$, $(0,0,1,0,0,4)$ and $(1,1,0,0,0,3)$. If we consider the transition from $(1,0,0,0,0,4)$ to $(2,0,0,0,0,3)$, we note that we start with a single forward vehicle and all other spots empty. The arrival rate is λ. There are nine empty spots and the probability of choosing a specific empty spot from an empty pair is double that of choosing the empty spot opposite the currently parked vehicle. Thus the probability of each of the 8 empty spots within empty pairs is 2/17 and the probability of the 1 empty spot opposite the parked vehicle is 1/17. So the rate of moving into a specific empty spot in an empty pair is $(2/17)\lambda$. By our assumptions, given that we choose such as spot, the probability of forward parking is .1 so the transition rate into a specific one of the eight spots is $(2/17).1\lambda$. Since there are eight such spots, the total transition rate from $(1,0,0,0,0,4)$ to $(2,0,0,0,0,3)$ is $(16/17).1\lambda$.

Similarly, we compute the rates from $(1,0,0,0,0,4)$ to $(0,0,0,1,0,4)$, $(0,0,1,0,0,4)$ and $(1,1,0,0,0,3)$ as $\frac{1}{17}*0.8\lambda$, $\frac{1}{17}*0.2\lambda$, $\frac{16}{17}*0.9\lambda$, respectively. See Table 3.

Table 3: Transition rates from $(1,0,0,0,0,4)$

From	To	Rate
$(1,0,0,0,0,4)$	$(0,0,0,0,0,5)$	μ
$(1,0,0,0,0,4)$	$(2,0,0,0,0,3)$	$\frac{16}{17}0.1\lambda$
$(1,0,0,0,0,4)$	$(0,0,0,1,0,4)3$	$\frac{1}{17}*0.8\lambda$
$(1,0,0,0,0,4)$	$(0,0,1,0,0,4)4$	$\frac{1}{17}*0.2\lambda$
$(1,0,0,0,0,4)$	$(1,1,0,0,0,3)5$	$\frac{16}{17}*0.9\lambda$

These rates would be the positive values corresponding to the row of the 252×252 matrix transitioning from state $(1, 0, 0, 0, 0, 4)$. Even though it is possible to write a infinitesimal generator matrix for these 252 cases, we choose to use simulation because of its large size and complex calculation.

3   Pseudo Code

Our pseudo code is shown below:

Algorithm 1 Algorithm for getting relation among parking direction and arrival time, service time.

Require: Arrival time parameter,λ;

 1: **function** MYFUNCTION(i)
 2:      $\mu \leftarrow 1$, $Time \leftarrow 0$
 3:      A is initialized to all zeros
 4:      **repeat**
 5:          Generate the time point t for the next event, from an exponential distribution with parameter $\dfrac{\lambda}{\lambda + \mu * sum(abs(A))}$.
 6:          Generate a random number r to determine whether the next event is arrival or departure. If $r < \dfrac{\lambda}{\lambda + \mu * sum(abs(A))}$, the next event is arrival; else it is a departure.
 7:          **if** the next event is arrival **then**
 8:              Create matrix B for parking probability of each space.
 9:              Generate random number r to select the parking space.
10:              Set matrix P for probability of parking forward at each space.
11:              Generate random number r to decide if parking is forward.
12:          **else** Generate random number r to select the parking space from which the vehicle leaves.
13:          **end if**
14:          Store the proportion of reverse parking in vector v.
15:          Time=Time+t.
16:      **until** $Time > 24$
17:      Select the last entry of the the vector v as *prop*.
18:      **return** *prop*.
19: **end function**

In order to create an algorithm with fast operating speed, we take advanatge of exponential distribution properties. We begin with our 5×2 matrix A initially with all zeros.

- We use the *memoryless property* of the exponential distribution and **Theorem** 1.1 to generate the time until the next event which is exponentially distributed with parameter $\lambda + \mu * sum(|A|)$. A more general but more complex method would begin by generating two sequences of interarrival times and service times.

- According to **Theorem** 1.2, we decide the type (arrival or service completion) of the next event by generating a uniform (0,1) random number r and comparing it to $\frac{\lambda}{\lambda + \mu * sum(|A|)}$.

4 Simulation and Results

We assume that λ vehicles arrive per hour, the average parking time per vehicle is $\mu = 1$ hour. The important value here is the ratio of λ to μ so we can choose $\mu = 1$ without loss of generality by simply using a different choice of time units.

Let F represent the number of forward facing vehicles and B represent the number of backward facing vehicles at the end of a single simulation run. After using Monte Carlo simulation for the parking lot with the above code, we get the proportion of backward parking as a function of λ shown below, where $E(\frac{B}{F+B})$ represents for the average of the proportion of reverse parking in all parking vehicles. See Figure 5. This assumes that the system is in steady state, so that the time measured from the start time is large.
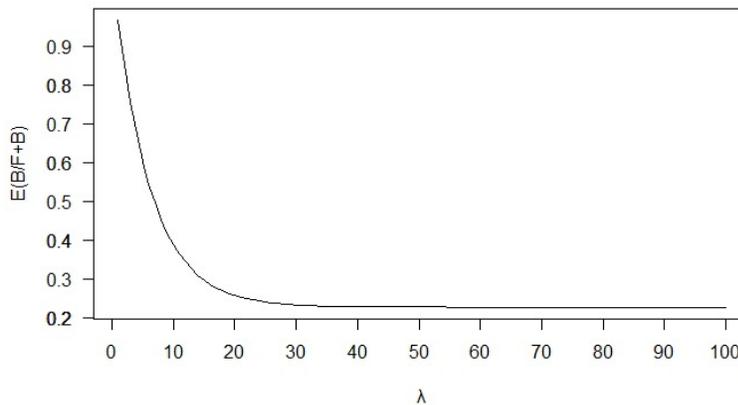

Figure 5: Proportion of Reverse Parking vs Lambda

From this image we can see that our oriented model is reasonable. When λ is close to 0, vehicles arrive at a slow rate which makes the parking lot often

almost empty, which means the drivers can drive through directly so that the vehicle is automatically in a "reverse" status or drivers are more willing to spend some time and effort to reverse park, as there is likely little interference from other vehicles. Thus $E(\frac{B}{F+B})$ approaches 0.9 (which is our choice of the reverse probability parameter when the both components of a pair are empty). When the λ is large, vehicles arrive very frequently to make the parking lot often full, which means the drivers prefer to park forward because the parking lot is busy and crowded. i.e. $E(\frac{B}{F+B})$ approaches 0.2 (which is our choice of the reverse parking probability parameter when one side of a pair is already full).

To illustrate further, we counted the four most frequent states when $\lambda = 2$, $\lambda = 5$, $\lambda = 8$, $\lambda = 10$. Recall that a state has six components representing the counts for the six cases Single Forward, Single Backward, One Forward and One Backward, Double Forward, Double Backward, or Empty.

Table 4: The Five Highest Frequency states

λ	$1st$	$2nd$	$3rd$	$4th$
$\lambda = 2$	$(0, 2, 0, 0, 0, 3)$	$(0, 4, 0, 0, 0, 1)$	$(0, 4, 1, 0, 0, 0)$	$(0, 3, 0, 0, 0, 2)$
$\lambda = 5$	$(0, 2, 0, 0, 0, 3)$	$(0, 4, 1, 0, 0, 0)$	$(0, 2, 0, 1, 0, 2)$	$(1, 2, 0, 0, 0, 2)$
$\lambda = 8$	$(0, 2, 1, 0, 0, 2)$	$(0, 3, 0, 0, 1, 1)$	$(1, 0, 0, 0, 0, 4)$	$(0, 3, 1, 0, 1, 0)$
$\lambda = 10$	$(0, 0, 4, 1, 0, 0)$	$(0, 0, 3, 2, 0, 0)$	$(0, 1, 3, 1, 0, 0)$	$(0, 1, 4, 0, 0, 0)$

From the table we can see that as λ increases, that is, the number of vehicles entering the parking lot per hour increases, more Double Forward counts appear in the parking lot.

We indicated that Figure 5 represents the proportion of reverse parking for the system in steady state. To numerically justify that claim, we consider the proportion as a function of time for several values of λ.
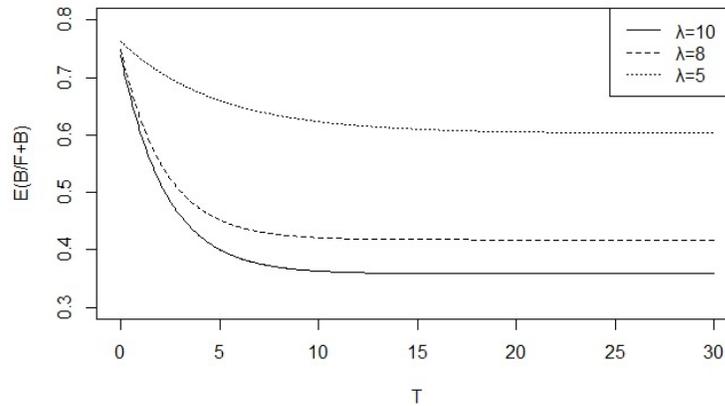
Figure 6: Proportion vs Time

Figure 6 depicts $E(\frac{B}{F+B})$ as a function of time when $\lambda = 5, 8, 10$. These three curves tend to level off to values $0.62, 0.42, 0.38$, beyond $T = 10$. These values are consistent with the values that appear in Figure 2 for the 3 values of λ used. Further, we notice that when the lambda is larger, the time it takes for the parking lot reverse parking proportion to stabilize will be shorter.

5   Conclusion and Future Research

According to our model, we find that the proportion of reverse parking for parked vehicles is related to the arrival rate and sojourn time of the vehicle. The larger the arrival rate, the lower the proportion of reverse parking will be. This model can help to explain the observed proportions of reverse parked vehicles in parking lots. If it is considered desirable to have a higher proportion of reverse parked vehicles, our model may suggest ways to achieve this. For example, more parking spots may mean a higher proportion of empty pairs in the lot which would encourage drivers to use reverse parking. Overall, our model can help the parking lot to develop a parking policy based on different vehicle entry and exit rates.

Further research on this topic is very possible. Our choice of probability values implies that the two possible access lanes are equally probable. In many cases, one access lane may be more convenient so probabilities could be adjusted to reflect that condition. Further, arrival rates will often change during the course of the day, so time dependent arrival rates could be added to the model to give more realistic results in many settings.

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