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A Class of Multi-Scales Nonlinear Difference Equations

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Abstract

In this paper, we give an iteration algorithm to compute asymptotic solutions for a class of nonlinear difference equations containing small parameters of multiple scales. We consider two kinds of perturbations.

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1 Introduction

Applied mathematicians and control scientists deal with practical systems involving nonlinear difference equations with small parameters of multiple time scales. The different scales arrange the convenience to reduce order and separate time-scale by using the singular perturbation methodology to reduce the complexity of these systems. Recently in [9], we developed an iterative method that gives asymptotic solutions for difference equations with one parameter, this procedure was initially introduced in various linear problems, see [3, 4, 5, 6, 7, 8]. In the present paper, we extend this procedure for a class of nonlinear equations containing several small multiscale parameters and we also allow variations for the boundary values. Let $(E, \|.\|)$ be a Banach space, $E_k \subset E$ be closed bounded, non-empty sets such that $E_{k+1} \subset E_k$. Let $\mathcal{U}_{l,k} : E_0 \times \cdots \times E_r \longrightarrow E$, $\mathcal{A}_k : E_0 \times \cdots \times E_{r-1} \longrightarrow E$ be p-differentiable functions, we consider the multi-scale difference equations

$$\begin{gathered}\textstyle\sum_{j=1}^{m}\varepsilon^{j}\mathcal{U}_{j,k}\left(x_k,\cdots,x_{k+n+j}\right)+\mathcal{A}_k\left(x_k,\cdots,x_{k+n}\right)=0,\\ 0\le k\le N-n-m,\end{gathered}\tag{1}$$

said of *left end perturbation*, satisfying the boundary conditions

$$x_k=\alpha_k(\varepsilon),\ k=0,\cdots,n-1,\quad x_{N-k}=\beta_k(\varepsilon),\ k=0,\cdots,m-1.\tag{2}$$

We assume that for $|\varepsilon|\le\delta<1$, $\alpha_k(\varepsilon)$ and $\beta_k(\varepsilon)$ have the asymptotic representations

$$\alpha_k(\varepsilon)=\alpha_k^{(0)}+\varepsilon\alpha_k^{(1)}+\cdots+\varepsilon^p\alpha_k^{(p)},\quad \beta_k(\varepsilon)=\beta_k^{(0)}+\varepsilon\beta_k^{(1)}+\cdots+\varepsilon^p\beta_k^{(p)}.\tag{3}$$

The linear case of BVP $(1)-(2)$ is studied in [4]; in the next Section we prove under suitable assumptions, the existence and uniqueness of a solution $x_k(\varepsilon)$, and we describe how to compute the coefficients of the asymptotic development

$$x_k(\varepsilon)=x_k^{(0)}+\varepsilon x_k^{(1)}+\varepsilon^2x_k^{(2)}+\cdots+\varepsilon^p x_k^{(p)}+\mathcal{O}(\varepsilon^{p+1}).\tag{4}$$

In Section 3, the method is promptly extended to difference equations with a *right-end perturbation*.

2 Main Results

2.1 Reduced Problem

Cancelling the parameter ε, the order of the difference equation in (1) drop to n providing the *reduced problem*

$$\begin{gathered}x_k=\alpha_k^{(0)},\quad k=0,\cdots,n-1,\\ \mathcal{A}_k\left(x_k,\cdots,x_{k+n}\right)=0,\quad 0\le k\le N-n-m,\end{gathered}\tag{5}$$

$$x_{N-k}=\beta_k^{(0)},\quad k=0,\cdots,m-1,\tag{6}$$

with uncoupled boundary conditions, the values $x_0,\ldots,x_{N-m}$ can be recursively computed from the IVP (5) without needing the final conditions; pursuant to the singular perturbation theory of ODES, it is stated as *singular perturbation* and there is a *boundary layer behavior* at (6).

Hypothesis 1. Suppose $D_n\mathcal{A}_k(x_k^{(0)},\cdots,x_{k+n-1}^{(0)},x)\neq 0$, for all $x\in E_n$, and the ranges of the functions $x\mapsto\mathcal{A}_k(x_k^{(0)},\cdots,x_{k+n-1}^{(0)},x)$, include zero.

Proposition 2.1 *If hypothesis 1 holds, then problem $(5)-(6)$ has a unique solution.*

2.2 Preliminaries

It is an asset that the BVP (1)−(2) may be reconsidered as a system of equations depending on a parameter. Let $X = (x_0, \ldots, x_N)$, we introduce the function $\mathcal{F}(\varepsilon, X) = (\mathcal{F}_0(\varepsilon, X), \ldots, \mathcal{F}_N(\varepsilon, X))$, where

$$\begin{gathered} \mathcal{F}_k = x_k - \alpha_k(\varepsilon), \quad k = 0, \cdots, n-1, \\ \mathcal{F}_{k+n} = \sum_{j=1}^{m} \varepsilon^j \mathcal{U}_{j,k}(x_k, \cdots, x_{k+n+j}) + \mathcal{A}_k(x_k, \cdots, x_{k+n}), \\ k = 0, \cdots, N-n-m, \\ \mathcal{F}_{N-k} = x_{N-k} - \beta_k(\varepsilon), \quad k = 0, \cdots, m-1, \end{gathered}$$

BVP (1)−(2) is equivalent to $\mathcal{F}(\varepsilon, x_0, x_1, \ldots, x_N) = 0$. By the Implicit Function Theorem [2], and under some assumptions, there exists a formal function of class C^p, we denote $g(\varepsilon) = (g_0(\varepsilon), \ldots, g_N(\varepsilon))$, such that $\mathcal{F}(\varepsilon, g(\varepsilon)) = 0$, or

$$\begin{gathered} \sum_{j=1}^{m} \varepsilon^j \mathcal{U}_{j,k}(g_k(\varepsilon), \cdots, g_{k+n+j}(\varepsilon)) + \mathcal{A}_k(g_k(\varepsilon) \ldots, g_{k+n}(\varepsilon)) = 0, \\ k = 0, \ldots, N-n-m \\ g_k(\varepsilon) = \alpha_k(\varepsilon),\ k = 0, \cdots, n-1, \quad g_{N-k}(\varepsilon) = \beta_k(\varepsilon),\ k = 0, \cdots, m-1. \end{gathered} \tag{7}$$

To find the coefficients of the Taylor polynomial expansion

$$g_k(\varepsilon) = g_k(0) + \varepsilon \frac{\dot{g}_k(0)}{1!} + \varepsilon^2 \frac{\ddot{g}_k(0)}{2!} + \cdots + \varepsilon^p \frac{g_k^{(p)}(0)}{p!} + \mathcal{O}(\varepsilon^{p+1}), \tag{8}$$

we compute the sequential derivatives of (7) and we use the Faà Di Bruno Formula [1]. To give concise formulas, we drop the arguments and we denote the partial derivative $\frac{\partial^k f(x_0, x_1, \ldots, x_n)}{\partial x_0^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}}$ by $D_0^{k_0} D_1^{k_1} \ldots D_n^{k_n} f$.

Lemma 2.2 *Assume that the functions g_k, $\mathcal{U}_{j,k}$ and $\mathcal{A}_k$ satisfy (7), and that all the necessary derivatives are defined. Then we have for $p \geq m$,*

$$\begin{gathered} \sum_{l=0}^{n} D_l \mathcal{A}_k g_{k+l}^{(p)} = -\underline{\sum_0 \cdots \sum_p} \frac{p! D_0^{p_0^{(0)}} \cdots D_n^{p_n^{(0)}} \mathcal{A}_k \prod_{i=1}^{p} (g_k^{(i)})^{q_{i0}^{(0)}} \cdots (g_{k+n}^{(i)})^{q_{in}^{(0)}}}{\prod_{i=1}^{p} (i!)^{k_i^{(0)}} \prod_{i=1}^{p} \prod_{j=0}^{n} q_{ij}^{(0)}!} - \\ \sum_0 \cdots \sum_{p-1} \frac{\binom{p}{1} 1!(p-1)! D_0^{p_0^{(1)}} \ldots D_{n+1}^{p_{n+1}^{(1)}} \mathcal{U}_{1,k} \prod_{i=1}^{p-1} (g_k^{(i)})^{q_{i0}^{(1)}} \cdots (g_{k+n+1}^{(i)})^{q_{in+1}^{(1)}}}{\prod_{i=1}^{p-1} (i!)^{k_i^{(1)}} \prod_{i=1}^{p-1} \prod_{j=0}^{n+1} q_{ij}^{(1)}!} - \cdots - \\ \sum_0 \cdots \sum_{p-m} \frac{\binom{p}{m} m!(p-m)! D_0^{p_0^{(m)}} \ldots D_{n+m}^{p_{n+m}^{(m)}} \mathcal{U}_{m,k} \prod_{i=1}^{p-m} (g_k^{(i)})^{q_{i0}^{(m)}} \cdots (g_{k+n+m}^{(i)})^{q_{in+m}^{(m)}}}{\prod_{i=1}^{p-m} (i!)^{k_i^{(m)}} \prod_{i=1}^{p-m} \prod_{j=0}^{n+m} q_{ij}^{(m)}!}. \end{gathered} \tag{9}$$

Agreeing that $g_k^{(p)} = 0$ for $p < 0$, the coefficients $k_i^{(l)}$, $q_{ij}^{(l)}$ and $p_j^{(l)}$, are all nonnegative integer solutions of the Diophantine equations

$$\begin{gathered} \sum_0 \to k_1^{(l)} + 2k_2^{(l)} + \cdots + (p-l)k_{p-l}^{(l)} = p - l,\ l = 0, \cdots, m, \\ \sum_i^l \to q_{i0}^{(l)} + q_{i1}^{(l)} + \cdots + q_{in+l}^{(l)} = k_i^{(l)},\ i = 1, \cdots, p-l,\ l = 0, \cdots, m, \\ p_j^{(l)} = q_{1j}^{(l)} + q_{2j}^{(l)} + \cdots + q_{p-l\,j}^{(l)},\ j = 0, \cdots, n+l,\ l = 0, \cdots, m, \\ k^{(l)} = p_0^{(l)} + p_1^{(l)} + \cdots + p_{n+l}^{(l)} = k_1^{(l)} + k_2^{(l)} + \cdots + k_{p-l}^{(l)},\ l = 0, \cdots, m, \end{gathered} \tag{10}$$

in $\underline{\sum_0 \cdots \sum_p}$ *we fix* $k_p^{(0)} = 0$*; the left side of (9) corresponds to* $k_p^{(0)} = 1$.

Proof 2.3 *By induction, we can easily prove for* $p \geq 1$,

$$\sum_{j=1}^{m} \sum_{l=0}^{j} \frac{\binom{p}{l} j! \varepsilon^{j-l}}{(j-l)!} \frac{d^{p-l}\mathcal{U}_{j,k}(g_k(\varepsilon), \cdots, g_{k+n+j}(\varepsilon))}{d\varepsilon^{p-l}} + \frac{d^p \mathcal{A}_k(g_k(\varepsilon), \cdots, g_{k+n}(\varepsilon))}{d\varepsilon^p} = 0, \tag{11}$$

we cancel the parameter ε *in (11), we obtain*

$$\sum_{j=1}^{m} \binom{p}{j} j! \frac{d^{p-j}\mathcal{U}_{j,k}(g_k(0), \cdots, g_{k+n+j}(0))}{(d\varepsilon)^{p-j}} + \frac{d^p \mathcal{A}_k(g_k(0), \cdots, g_{k+n}(0))}{d\varepsilon^p} = 0. \tag{12}$$

We find the formula (9) by expanding Faa Di Bruno Formula [1] into (12), and by arranging the equation so that on the left hand side, we have the terms corresponding to $k_p^{(0)} = 1$ *in the diophantine equations (10).*

2.3 Description of the Method

It is understood that the coefficients of 0^{th} order in (4) satisfy the *reduced* problem $(5)-(6)$. To determine the coefficients of higher order, we substitute

$$x_k^{(p)} = \frac{g_k^{(p)}(0)}{p!}, \quad k = 0, \cdots, N, \tag{13}$$

into (9), then 1^{st} order coefficients are defined by the iteration

$$\begin{array}{c} x_k^{(1)} = \alpha_k^{(1)},\ k = 0, \ldots, n-1, \\ D_n \mathcal{A}_k(x_k^{(0)}, \cdots, x_{k+n}^{(0)}) x_{k+n}^{(1)} = -\sum_{l=0}^{n-1} D_l \mathcal{A}_k(x_k^{(0)}, \cdots, x_{k+n}^{(0)}) x_{k+l}^{(1)} \\ -\mathcal{U}_k(x_k^{(0)}, \cdots, x_{k+n+1}^{(0)}),\ k = 0, \ldots, N-n-m, \\ x_{N-k}^{(1)} = \beta_k^{(1)},\ k = 0, \ldots, m-1. \end{array} \tag{14}$$

which starts from the initial values without needing the final conditions. The coefficents of 2^{nd} order satisfy

$$\begin{array}{c} x_k^{(2)} = \alpha_k^{(2)}, \quad k = 0, \ldots, n-1, \\ D_n \mathcal{A}_k x_{k+n}^{(2)} = -\sum_{l=0}^{n-1} D_l \mathcal{A}_k x_{k+l}^{(2)} - \sum_{0 \leq l < r \leq n} D_l D_r \mathcal{A}_k x_{k+l}^{(1)} x_{k+r}^{(1)} \\ -\frac{1}{2!} \sum_{l=0}^{n} D_{1+l}^2 \mathcal{A}_k \left(x_{k+l}^{(1)}\right)^2 - \sum_{l=0}^{n+1} D_l \mathcal{U}_{1,k} x_{k+l}^{(1)} - \mathcal{U}_{2,k}(x_k^{(0)}, \cdots, x_{k+n+2}^{(0)}), \\ k = 0, \ldots, N-n-m, \\ x_{N-k}^{(2)} = \beta_k^{(2)} \quad k = 0, \ldots, m-1, \end{array} \tag{15}$$

the arguments are removed to give shorter formulas, the coefficients are computed recursively using the initial values and the 1^{st} order solution determined in the preceding stage. In general, agreeing that $x_k^{(p)} = 0$ for $p < 0$, we have

$$\begin{aligned}
&x_0^{(p)} = \alpha_0^{(p)}, \quad x_1^{(p)} = \alpha_1^{(p)}, \quad \cdots \quad , \quad x_{n-1}^{(p)} = \alpha_{n-1}^{(p)}, \\
&D_n \mathcal{A}_k x_{k+n}^{(p)} = -\sum\nolimits_0 \cdots \sum\nolimits_{p-1} \frac{D_0^{p_0^{(1)}} \dots D_{n+1}^{p_{n+1}^{(1)}} \mathcal{U}_{1,k} \prod_{i=1}^{p-1} (x_k^{(i)})^{q_{i0}^{(1)}} \cdots (x_{k+n+1}^{(i)})^{q_{in+1}^{(1)}}}{\prod_{i=1}^{p-1} \prod_{j=0}^{n+1} q_{ij}^{(1)}!} \\
&- \cdots - \sum\nolimits_0 \cdots \sum\nolimits_{p-m} \frac{D_0^{p_0^{(m)}} \dots D_{n+m}^{p_{n+m}^{(m)}} \mathcal{U}_{m,k} \prod_{i=1}^{p-m} (x_k^{(i)})^{q_{i0}^{(m)}} \cdots (x_{k+n+m}^{(i)})^{q_{in+m}^{(m)}}}{\prod_{i=1}^{p-m} \prod_{j=0}^{n+m} q_{ij}^{(m)}!} \\
&- \underline{\sum\nolimits_0 \cdots \sum\nolimits_p} \frac{D_0^{p_0^{(0)}} \dots D_n^{p_n^{(0)}} \mathcal{A}_k \prod_{i=1}^{p} (x_k^{(i)})^{q_{i0}^{(0)}} \cdots (x_{k+n}^{(i)})^{q_{in}^{(0)}}}{\prod_{i=1}^{p} \prod_{j=0}^{n} q_{ij}^{(0)}!} - \sum\nolimits_{l=0}^{n-1} D_l \mathcal{A}_k x_{k+l}^{(p)},
\end{aligned} \tag{16}$$

the initial values are used in the iteration process while the final value are fixed

$$x_{N-k}^{(p)} = \beta_k^{(p)} \quad k = 0, \dots, m-1. \tag{17}$$

Algorithm. After completing the coefficient calculations, we replace in (4) then we find the desired p^{th} order approximate solution for the BVP (1)$-$(2). The process is validated in the following theorem.

Theorem 2.4 *If hypothesis 1 holds, then there exists $\epsilon > 0$, such that for all $|\varepsilon| < \epsilon$, the BVP (1)$-$(2) has a unique solution $(x_k(\varepsilon))_{k=0,\cdots,N}$ satisfying (4); the coefficients $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, $x_k^{(n)}$, are the solutions of the problems (5)$-$(6), (14), (15), (16)$-$(17), respectively.*

Proof 2.5 *We denote: $\tilde{X} := (\varepsilon, X)$, $\widehat{\mathcal{F}}\left(\tilde{X}\right) = (\varepsilon, \mathcal{F}(\varepsilon, X))$ and $\mathcal{D}\widehat{\mathcal{F}}$ its Jacobian. From **H 1** we deduce that $\widehat{\mathcal{F}}$ is locally invertible since the determinant of its Jacobian at $\tilde{X}^{(0)}$ is equal to $\prod_{i=0}^{N-n-m} \mathcal{D}_n \mathcal{A}_i \left(x_i^{(0)}, \cdots, x_{i+n}^{(0)}\right)$ which does not cancel. Because $\mathcal{D}\widehat{\mathcal{F}}$ is continuous, we have*

$$\exists \rho > 0, \forall \tilde{X} \in B\left(\tilde{X}^{(0)}, \rho\right) : \|\mathcal{D}\widehat{\mathcal{F}}\left(\tilde{X}\right) - \mathcal{D}\widehat{\mathcal{F}}\left(\tilde{X}^{(0)}\right)\| < \frac{1}{2\|\mathcal{D}\widehat{\mathcal{F}}(\tilde{X}^{(0)})^{-1}\|}.$$

Let $\epsilon = \frac{\rho}{2\|\mathcal{D}\widehat{\mathcal{F}}(\tilde{X}^{(0)})^{-1}\|}$, and $G_Y(\tilde{X}) = \tilde{X} - \|\mathcal{D}\widehat{\mathcal{F}}(\tilde{X}^{(0)})^{-1}\| \left(\widehat{\mathcal{F}}(\tilde{X}) - Y\right)$. For $\|Y\| < \epsilon$, we verify that G_Y is a contraction from $B\left(\tilde{X}^{(0)}, \rho\right)$ to itself, so G_Y has a unique fixed point. Therefore, there is a unique $\tilde{X}$ in $B\left(\tilde{X}^{(0)}, \rho\right)$, such that $Y = \widehat{\mathcal{F}}(\tilde{X})$. Let $g(\varepsilon) = (g_0(\varepsilon), \cdots, g_N(\varepsilon))$, if $|\varepsilon| < \epsilon$, then for $(\varepsilon, 0, \cdots, 0) \in B(0, \epsilon)$, there is a unique $(\varepsilon, g(\varepsilon))$ in $B\left(\tilde{X}^{(0)}, \rho\right)$, such that $(\varepsilon, 0, \cdots, 0) = \widehat{\mathcal{F}}(\varepsilon, g(\varepsilon))$. It means that the BVP (1)$-$(2) has a unique solution for $|\varepsilon| < \epsilon$. Moreover g is $C^p(-\epsilon, \epsilon)$ as are $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}^{-1}$. By the chain rule *we have* $\dot{g}(\varepsilon) = -\frac{\partial \mathcal{F}(\varepsilon, g(\varepsilon))}{\partial \varepsilon}\left(\frac{\partial \mathcal{F}(\varepsilon, g(\varepsilon))}{\partial X}\right)^{-1}$, *higher derivatives of* g *are given in Lemma 2.2.*

Hypothesis 2. We assume that $\|\alpha_k^{(i)}\| \leq \frac{A}{\delta^k}$, $\|\beta^{(i)}\| \leq \frac{B}{\delta^k}$, A and B are constants, the functions $\mathcal{U}_k$ and $\mathcal{A}_k$ are differentiable at any order (smooth).

Theorem 2.6 *If Hypothesis 2 holds, there exists* $\epsilon > 0$, *such that for all* $|\varepsilon| < \epsilon$, *the BVP (1)–(2) has a unique solution* $(x_k(\varepsilon))_{k=0,\cdots,N}$ *satisfying* $x_k(\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p x_k^{(p)}$, *where* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, $x_k^{(p)}$ *are the solutions of the problems (5)–(6), (14), (15), (16)–(17), respectively.*

3 Right End Perturbation

The results obtained in Section 2 can be easily extended to equations presenting a *right-end perturbation.* Consider the boundary value problem

$$\begin{gathered}\sum_{j=1}^{m} \varepsilon^j \mathcal{U}_{j,k}\left(x_k, \cdots, x_{k+n+j}\right) + \mathcal{A}_k\left(x_{k+n}, \cdots, x_{k+n+m}\right) = 0, \\ 0 \leq k \leq N-n-m\end{gathered} \tag{18}$$

$$x_k = \alpha_k(\varepsilon), \quad k = 0, \cdots, n-1, \quad x_{N-k} = \beta_k(\varepsilon), \quad k = 0, \cdots, m-1. \tag{19}$$

The linear case of (18)–(19) is studied in [3, 4]. In an analogous way to the method developed for *left-end perturbation*, similar results linked to (18)–(19) may be obtained without difficulties. Deleting the parameter in (18), follows

$$x_k = \alpha_k^{(0)}, \quad k = 0, \cdots, n-1, \tag{20}$$

$$\begin{gathered}\mathcal{A}_k\left(x_{k+n}, \cdots, x_{k+n+m}\right) = 0, \quad 0 \leq k \leq N-n-m, \\ x_{N-k} = \beta_k^{(0)}, \quad k = 0, \cdots, m-1.\end{gathered} \tag{21}$$

The *boundary layer behavior* is located at the initial values (20), to solve (21) we compute backward using the final values.
Hypothesis 3. Suppose that $D_0\mathcal{A}_k(x, x_{k+2}^{(0)}, \cdots, x_{k+n+m}^{(0)}) \neq 0 \ \forall x \in E_{k+n}$, and the ranges of the functions $x \mapsto \mathcal{A}_k\left(x, x_{k+n+1}^{(0)}, \cdots, x_{k+n+m}^{(0)}\right)$ contain 0.

Proposition 3.1 *If hypothesis 3 holds, then problem (21) has a unique solution.*

By the Implicit Function Theorem, we can find under some assumptions, a function $g(\varepsilon) = (g_0(\varepsilon), \cdots, g_N(\varepsilon))$, such that

$$\begin{gathered}\sum_{j=1}^{m} \varepsilon^j \mathcal{U}_{j,k}\left(g_k(\varepsilon), \cdots, g_{k+n+j}(\varepsilon)\right) + \mathcal{A}_k\left(g_{k+n}(\varepsilon), \cdots, g_{k+n+m}(\varepsilon)\right) = 0, \\ k = 0, \ldots, N-n-m, \\ g_k = \alpha_k(\varepsilon), \quad k = 0, \cdots, n-1, \quad g_{N-k}(\varepsilon) = \beta_k, \quad k = 0, \cdots, m-1.\end{gathered} \tag{22}$$

Lemma 3.2 *Assume that the functions g_k, $\mathcal{U}_k$ and $\mathcal{A}_k$ satisfy (22), and that all the necessary derivatives are defined. Then we have for $n \geq 2$,*

$$\begin{array}{c} \sum_{l=0}^{m} D_l \mathcal{A}_k g_{k+n+l}^{(p)} = -\underline{\sum_0 \cdots \sum_p} \frac{p! D_0^{p_0^{(0)}} \cdots D_m^{p_m^{(0)}} \mathcal{A}_k \prod_{i=1}^{p} (g_{k+n}^{(i)})^{q_{i0}^{(0)}} \cdots (g_{k+n+m}^{(i)})^{q_{in}^{(0)}}}{\prod_{i=1}^{p} (i!)^{k_i^{(0)}} \prod_{i=1}^{p} \prod_{j=0}^{m} q_{ij}^{(0)}!} \\ -\sum_0 \cdots \sum_{p-1} \frac{\binom{p}{1} 1!(p-1)! D_0^{p_0^{(1)}} \cdots D_{n+1}^{p_{n+1}^{(1)}} \mathcal{U}_{1,k} \prod_{i=1}^{p-1} (g_k^{(i)})^{q_{i0}^{(1)}} \cdots (g_{k+n+1}^{(i)})^{q_{in+1}^{(1)}}}{\prod_{i=1}^{p-1} (i!)^{k_i^{(1)}} \prod_{i=1}^{p-1} \prod_{j=0}^{n+1} q_{ij}^{(1)}!} - \cdots \\ -\sum_0 \cdots \sum_{p-m} \frac{\binom{p}{m} m!(p-m)! D_0^{p_0^{(m)}} \cdots D_{n+m}^{p_{n+m}^{(m)}} \mathcal{U}_{m,k}}{\prod_{i=1}^{p-m} (i!)^{k_i^{(m)}} \prod_{i=1}^{p-m} \prod_{j=0}^{n+m} q_{ij}^{(m)}!} \prod_{i=1}^{p-m} (g_k^{(i)})^{q_{i0}^{(m)}} \cdots (g_{k+n+m}^{(i)})^{q_{in+m}^{(m)}}. \end{array} \tag{23}$$

Agreeing that $g_k^{(p)} = 0$ for $p < 0$, the coefficients $k_i^{(l)}$, $q_{ij}^{(l)}$ and $p_j^{(l)}$, are all nonnegative integer solutions of the Diophantine equations

$$\begin{array}{c} \sum_0 \rightarrow k_1^{(l)} + 2k_2^{(l)} + \cdots + (p-l)k_{p-l}^{(l)} = p - l, \quad l = 0, \cdots, m, \\ \sum_i^0 \rightarrow q_{i0}^{(0)} + q_{i1}^{(0)} + \cdots + q_{im}^{(0)} = k_i^{(0)}, \quad i = 1, \cdots, p, \\ \sum_i^l \rightarrow q_{i0}^{(l)} + q_{i1}^{(l)} + \cdots + q_{i\,n+l}^{(l)} = k_i^{(l)}, \quad i = 1, \cdots, p-l, \quad l = 1, \cdots, m, \\ p_j^{(l)} = q_{1j}^{(l)} + q_{2j}^{(l)} + \cdots + q_{p-l\,j}^{(l)}, \quad l = 0, \cdots, m, \end{array} \tag{24}$$

in $\underline{\sum_0 \cdots \sum_p}$ we fix $k_p^{(0)} = 0$; the case $k_p^{(0)} = 1$ is omitted and corresponds to the left side of equation (23).

We can already indicate the main result of this section. From (8), (13) and (23), we deduce that the coefficients of the 1^{st} order development, satisfy

$$\begin{array}{c} x_k^{(1)} = \alpha_k^{(1)}, \quad k = 0 \cdots, n-1, \\ D_0 \mathcal{A}_k(x_{k+n}^{(0)}, \cdots, x_{k+n+m}^{(0)}) x_{k+n}^{(1)} = -\mathcal{U}_{1,k}(x_k^{(0)}, \cdots, x_{k+n+1}^{(0)}) \\ -\sum_{l=1}^{m} D_l \mathcal{A}_k(x_{k+n}^{(0)}, \cdots, x_{k+n+m}^{(0)}) x_{k+n+l}^{(1)}, \quad k = 0, \ldots, N-n-m, \\ x_{N-k}^{(1)} = \beta_k^{(1)}, \quad k = 0, \cdots, m-1. \end{array} \tag{25}$$

The coefficients $x_n^{(1)}, x_{n+1}^{(1)}, \cdots, x_{N-m}^{(1)}$ are calculated backward from the $m-1$ final values regardless of the initial value which are fixed at 0; the 0^{th} order solution is needed. In what follows, the arguments are removed. For 2^{nd} order development, we have the iterative process

$$\begin{array}{c} x_k^{(2)} = \alpha_k^{(2)}, \ k = 0 \cdots, n-1, \\ D_0 \mathcal{A}_k(x_{k+n}^{(0)}, \cdots, x_{k+n+m}^{(0)}) x_{k+n}^{(2)} = -\sum_{l=1}^{m} D_l \mathcal{A}_k(x_{k+n}^{(0)}, \cdots, x_{k+n+m}^{(0)}) x_{k+n+l}^{(2)} \\ -\sum_{0 \leq l < r \leq m} D_l D_r \mathcal{A}_k x_{k+n+l}^{(1)} x_{k+n+r}^{(1)} - \frac{1}{2!} \sum_{l=0}^{m} D_l^2 \mathcal{A}_k \left(x_{k+n+l}^{(1)} \right)^2 \\ -\mathcal{U}_{2,k}(x_k^{(0)}, \cdots, x_{k+n+2}^{(0)}), \ k = 0, \ldots, N-n-m, \\ x_{N-k}^{(2)} = \beta_k^{(2)}, \ k = 0, \cdots, m-1. \end{array} \tag{26}$$

For pth order development, $p \geq 2$, agreeing that $x_k^{(p)} = 0$ for $p < 0$, we have

$$
\begin{aligned}
x_k^{(p)} &= \alpha_k^{(p)}, \quad k = 0 \cdots, n-1, \\
\sum_{l=0}^{m} D_l \mathcal{A}_k x_{k+n+l}^{(p)} &= -\underline{\sum_0 \cdots \sum_p} \frac{D_0^{p_0^{(0)}} \cdots D_m^{p_m^{(0)}} \mathcal{A}_k \prod_{i=1}^{p} (x_{k+n}^{(i)})^{q_{i0}^{(0)}} \cdots (x_{k+n+m}^{(i)})^{q_{in}^{(0)}}}{\prod_{i=1}^{p} \prod_{j=0}^{m} q_{ij}^{(0)}!} \\
&\quad - \sum_0 \cdots \sum_{p-1} \frac{D_0^{p_0^{(1)}} \cdots D_{n+1}^{p_{n+1}^{(1)}} \mathcal{U}_{1,k} \prod_{i=1}^{p-1} (x_k^{(i)})^{q_{i0}^{(1)}} \cdots (x_{k+n+1}^{(i)})^{q_{in+1}^{(1)}}}{\prod_{i=1}^{p-1} \prod_{j=0}^{n+1} q_{ij}^{(1)}!} - \cdots \\
&\quad - \sum_0 \cdots \sum_{p-m} \frac{D_0^{p_0^{(m)}} \cdots D_{n+m}^{p_{n+m}^{(m)}} \mathcal{U}_{m,k} \prod_{i=1}^{p-m} (x_k^{(i)})^{q_{i0}^{(m)}} \cdots (x_{k+n+m}^{(i)})^{q_{in+m}^{(m)}}}{\prod_{i=1}^{p-m} \prod_{j=0}^{n+m} q_{ij}^{(m)}!}, \\
x_{N-k}^{(p)} &= \beta_k^{(p)}, \quad k = 0, \cdots, m-1,
\end{aligned}
\tag{27}
$$

the computations are done backward from the final values while the initial value remains fixed. The proof of the following theorems are left to the reader.

Theorem 3.3 *If Hypothesis 3 holds, then* $\exists\ \epsilon > 0$, $\forall\ |\varepsilon| < \epsilon$, *the BVP (18)–(19) has a unique solution* $(x_k(\varepsilon))_{k=0,\cdots,N}$, *which satisfy (4), where* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, $x_k^{(p)}$, *are the solutions of (20)–(21), (25), (26), (27), respectively.*

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Theorem 3.4 *If Hypothesis 2 holds,* $\exists\ \epsilon > 0$, *such that* $\forall\ |\varepsilon| < \epsilon$, *the BVP 18–19 has a unique solution* $(x_k(\varepsilon))_{k=0,\cdots,N}$, *which satisfy* $x_k(\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p x_k^{(p)}$ *where* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, $x_k^{(p)}$, *are solutions of (20)–(21), (25), (26), (27), respectively.*

4 Conclusion

As the results show, the theory of singular perturbation for difference equations includes the same list of ingredients as for the singular perturbation theory for ODEs, a separation of time scales, an order reduction, and boundary layer phenomena. Instead of singularly perturbed differential equations, we can find homogeneous development for singularly perturbed difference equations. Initial value problems on finite time interval could be treated by the same methods.

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