Applied Mathematical Sciences, Vol. 10, 2016, no. 21, 1023 - 1030
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http://dx.doi.org/10.12988/ams.2016.6133

Control Problem of a Deterministic Queuing System

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Abstract

In the paper the concept of the cycle of operation of a deterministic queuing system and the concept of the steady-state operation of the system is introduced. Deterministic queueing systems with two and three queues and one servicing device are considered. The server serves only one requirement at each given time.

A problem of optimal timing of switching from one device queue to another is studied.

Necessary and sufficient conditions for stationarity of functioning of a service system are obtained. Discussed problems close to the well-known problem of controlling of a traffic light in an isolated intersection.

Mathematics Subject Classification: 60P20

Keywords: Deterministic Queuing System, Service Cycle, Stationary Regime

1 Introduction

The article discusses the an unconventional formulation of the control problem for one system management. In the paper [2] the unconventional problem for other system was considered. Consider a deterministic queuing system which contains a single serving unit with three streams of applications. Speeds of receipt of applications as well as speeds of handling of applications by a service device depend on the quantity of the queue. At any moment the server can handle only one application. Service systems of such type have proliferated in recent years. For example, in various service centers an user of a terminal device chooses the queue number in accordance with the type of his application, then obtains a number in the chosen queue for service. The service comes with using a multifunctional operating device, which switches from one queue to another during an operation and wherein moments of switching are chosen by the service device. The formulated problem is similar to the well-known problem of the control of traffic lights at an isolated intersection ([1] - [10]), but significantly differs from it by the nature of the restrictions, in particular, it is generally assumed that the time of service in the problem of the intersection is equal to zero. Under the problem of managing such a system it is possible to understand the choice of the switching procedure of a servicing device with one queue to another, guaranteeing that there is no unlimited growth of the queue on each streams of applications. A similar problem with two streams was considered earlier in [9].

2 Preliminary Notes

The scheme of work of the service system under consideration is presented in Fig. 1.

Introduce the following notation: let $q_1(t)$, $q_2(t)$, $q_3(t)$ be queue lengths waiting for service of a multifunctional device for the first, second and third streams at the time t respectively. Let $a_i(t)$ $d_i(t)$ be speeds of receipt and fulfillment orders for the i-th line, respectively, where $i = 1, 2, 3$. Let g_i be the duration of continuous service of requests from the queue with the number i, $g_i > 0$ $(i = 1, 2, 3)$. Let's assume that:

1. $a_i(t) = a_i \geq 0$ is a known constant;

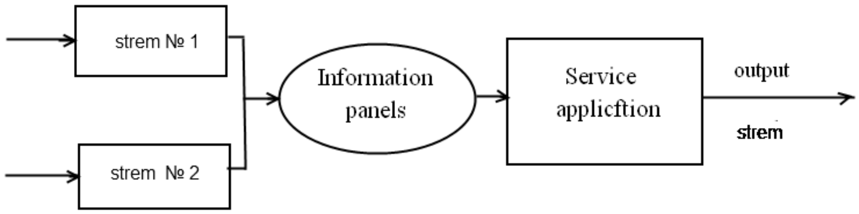

Fig. 1. One queuing system and three queues.

2. $q_i(t)$ is a non-negative integer (the number of requests in the queue for service flow i at time t).

3. $d_i(t) = \begin{cases} 0, & \text{if the device supports an application from the queue } j \neq i \text{ ;} \\ d_i, & \text{if the device supports an application from the queue } i \text{ ;} \end{cases}$

4. $d_i > a_i$, note that in the framework of our problem these quantities take constant values.

5. In the initial moment of time the queue is absent, i.e. $q_i(0) = 0, i = 1, 2, 3$.

6. Let the duration of continuous service requests from the same queue put the same for each of the queues.

Definition 2.1 *The triple* (g_1, g_2, g_3) *is called a cycle, where* g_i *is lengths of continuous service requests from the queue with the number i* $(i = 1, 2, 3)$*.*

Let us consider three sequences of time points. The first sequence:

$$\tau_1^{(1)} = g_1,\ \ \tau_2^{(1)} = g_1 + g_2 + g_3 + g_1,\ \ \ldots,\ \ \tau_{k+1}^{(1)} = (g_1 + g_2 + g_3)k + g_1,\ \ \ldots.$$

This sequence of time points represents the points of start of service requests from the queue with the number two or the time of termination of the implementation of the requirements of the first stream.

The second sequence:

$$\tau_1^{(2)} = g_1+g_2,\ \ \tau_2^{(2)} = g_1+g_2+g_3+g_1+g_2,\ \ \ldots,\ \ \tau_{k+1}^{(2)} = (g_1+g_2+g_3)k+g_1+g_2,\ \ \ldots$$

The second sequence of time points represents the points of start of service requests from the queue with the number three, or the time of termination of the implementation of the requirements of the second stream.

The third sequence:

$\tau_1^{(3)} = g_1+g_2+g_3, \tau_2^{(3)} = g_1+g_2+g_3+g_1+g_2+g_3, \ldots, \tau_{k+1}^{(3)} = (g_1+g_2+g_3)(k+1), ...$

The third sequence of time points represents points of start of service requests from the queue with the number one, or the time of termination of the implementation of the requirements of the third stream. Let's introduce a notation for the initial time: $\tau_0^{(0)} = 0$ is the start time of the MFD (a receiption of first request for service).

Definition 2.2 *Such regime of service applications in which there will be accumulation of the queue i.e., the following conditions*

$$q_1\left(\tau_{k+1}^{(1)}\right) = 0,\ q_2\left(\tau_{k+1}^{(2)}\right) = 0,\ q_3\left(\tau_{k+1}^{(3)}\right) = 0 \quad \forall k = 0, 1, 2, \ldots.$$

hold is called a stationary regime.

3 Necessary and sufficient conditions for stationarity of three queues

Now we find out conditions when the cycle (g_1, g_2, g_3) will lead to the stationary regime.

Theorem 3.1 *A cycle (g_1, g_2, g_3) generates a stationary regime if and only if when the following inequalities*

$$\frac{d_1 - a_1}{a_1} \geq \frac{g_2 + g_3}{g_1}; \quad \frac{d_2 - a_2}{a_2} \geq \frac{g_1 + g_3}{g_2}; \quad \frac{d_3 - a_3}{a_3} \geq \frac{g_1 + g_2}{g_3} \tag{1}$$

hold.

This theorem is proved similarly to the first theorem from [9]. As opposed to the theorem from [9] in this theorem the question of the existence of a stationary regime for a service system with characteristics of d_i, a_i $(i = 1, 2, 3, \ldots)$ is not obvious. The following theorem gives an answer of this question.

Theorem 3.2 *Let $q_1(0) = q_2(0) = q_3(0) = 0$. The cycle (g_1, g_2, g_3) generating a stationary regime exists if and only if when the following conditions*

$$\frac{d_1 - a_1}{a_1} > \frac{a_2}{d_2 - a_2}; \quad \frac{d_3 - a_3}{a_3} > \frac{a_2}{d_2 - a_2};$$
$$\frac{d_3 - a_3}{a_3} > \frac{a_1}{d_1 - a_1}; \quad \frac{d_3}{a_3} \geq \frac{d_1 d_2}{d_1 d_2 - d_1 a_2 - d_2 a_1} \tag{2}$$

hold.

Proof. We change variables

$$\frac{g_1}{g_3} = u_1; \quad \frac{g_2}{g_3} = u_2. \tag{3}$$

Note that new parameters u_1, u_2, are positive as the duration of the continuous service g_i which takes values greater than zero. Thus the necessary and sufficient conditions of Theorem 3.1 in the new parameterization are given by the following inequalities

$$u_2 + 1 \leq \frac{d_1 - a_1}{a_1} u_1; \quad u_1 + 1 \leq \frac{d_2 - a_2}{a_2} u_2; \quad u_1 + u_2 \leq \frac{d_3 - a_3}{a_3}. \tag{4}$$

In the rectangular coordinate system (u_1, u_2) we construct the graphs of the following functions

$$u_2 = \frac{d_1 - a_1}{a_1} u_1 - 1, \tag{5}$$

$$u_2 = \frac{a_2}{d_2 - a_2} + \frac{a_2}{d_2 - a_2} u_1, \tag{6}$$

$$u_2 = \frac{d_3 - a_3}{a_3} - u_1. \tag{7}$$

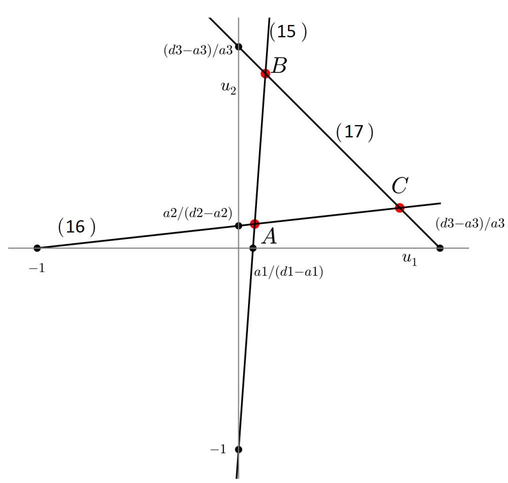

Fig. 2. The set of solutions of system of inequalities (4).

Find a solution of system (4). Note, we are interested in only the first quarter, as the parameters $u_1 > 0$, $u_2 > 0$. The solution is presented in Fig. 2.

This system of inequalities is not always the solution, thus we cannot always guarantee the establishment of the stationary regime. Let us derive necessary and sufficient conditions under which system inequalities (4) has no solution.

1. The slope of the straight line (5) must be greater than the slope of the straight line (6). In case of equality of slopes the graph of the function (5) is parallel to the graph of the function (6), and in a case where the slope of the (5) is less than the slope of the (6) these graphs do not intersect in the first quarter. In other words if the inequality

$$\frac{d_1 - a_1}{a_1} > \frac{a_2}{d_2 - a_2},$$

   does not hold then system (4) will not have solutions.

2.

$$\frac{a_2}{d_2 - a_2} < \frac{d_3 - a_3}{a_3}.$$

   The point of intersection of the graph of the function (6) with axis u_2 should not be above the point of intersection of the axis u_2 with the graph of (7). Otherwise, the system of inequalities (4) will not have a solution in the first quarter. In other words the following condition:

$$\frac{a_2}{d_2 - a_2} < \frac{d_3 - a_3}{a_3}.$$

   must be performed.

3. The point of intersection of the graph of (7) with axis u_1 must lie to the right of the point of intersection of the axis u_1 with the graph of (5). Otherwise the set of solutions of system of inequalities (4) will not satisfy the condition of nonnegativity of the parameters u_1, u_2. I.e. the inequality

$$\frac{a_1}{d_1 - a_1} < \frac{d_3 - a_3}{a_3}$$

   hold.

4. The intersection point of graphs (5) and (6) must lie below the line (7). Find a coordinate u_1 of the point of intersection. To do this, equate the coordinate u_2 from equations (5) and (6). We receive

$$\frac{d_1 - a_1}{a_1} u_1 - 1 = \frac{a_2}{d_2 - a_2} u_1 + \frac{a_2}{d_2 - a_2}.$$

   Expressing from this equation the coordinate u_1 we obtain the following equality

$$u_1^{(A)} = \frac{1 + \frac{a_2}{d_2 - a_2}}{\frac{d_1 - a_1}{a_1} - \frac{a_2}{d_2 - a_2}}.$$

After bringing similar terms we obtain

$$u_1^{(A)} = \frac{d_2 a_1}{d_2 d_1 - a_1 d_2 - a_2 d_1}.$$

That is, the inequality

$$\frac{d_1 - a_1}{a_1} \frac{d_2 a_1}{d_2 d_1 - d_2 a_1 - d_1 a_2} - 1 \leq \frac{d_3 - a_3}{a_3} - \frac{d_2 a_1}{d_2 d_1 - d_2 a_1 - d_2 a_2}$$

has to hold. After transformation the above condition takes the form

$$\frac{d_1 d_2}{d_1 d_2 - d_2 a_1 - d_1 a_2} \leq \frac{d_3}{a_3}.$$

Therefore we arrive to a system of inequalities:

$$\frac{d_1 - a_1}{a_1} > \frac{a_2}{d_2 - a_2}; \quad \frac{d_3 - a_3}{a_3} > \frac{a_2}{d_2 - a_2};$$

$$\frac{d_3 - a_3}{a_3} > \frac{a_1}{d_1 - a_1}; \quad \frac{d_3}{a_3} \geq \frac{d_1 d_2}{d_1 d_2 - d_1 a_2 - d_2 a_1}.$$

Note that these conditions were obtained by using identical transformations of conditions (1), which, as shown above, are necessary and sufficient conditions of establishment of the stationary regime in our service system. Thus fulfillment of conditions (2) also guarantee us fulfillment of conditions (1). This means that the cycle (g_1, g_2, g_3) generating a stationary regime exists if and only if conditions(2) hold. The theorem is proved.

4 Conclusion

In this paper the deterministic queuing system with three queues is investigated. Necessary and sufficient conditions for stationarity of the functioning of the deterministic queuing system are given.

Acknowledgements. The work is supported by the Saint Petersburg State University (project no. 9.38.205.2014).

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Received: February 9, 2016; Published: March 23, 2016