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Coupling Homogenization and Large Deviations Principle in a Parabolic PDE

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Abstract

We study the behaviour of the PDE solution if $\frac{\delta}{\varepsilon}$ tends to a positive constant $\gamma > 0$ when the two parameters δ (homogenization parameter) and ϵ (the large deviations parameter) tend to zero with the same speed.

Mathematics Subjection Classification: 60H15, 35K, 35B27

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Introduction

Consider the partial differential equation (PDE) given on $\mathbb{R}^d$ by :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = L_{\varepsilon,\delta}u^{\varepsilon,\delta}(t,x) + \frac{1}{\varepsilon} f\left(\frac{x}{\delta}, u^{\varepsilon,\delta}(t,x)\right) \\ \\ u^{\varepsilon,\delta}(0,x) = g(x), \qquad x \in \mathbb{R}^d \end{cases} \tag{1}$$

where f is a 1-periodic non-linear function such that :

- $\forall x \in \mathbb{R}^d,\ f(x,1) = 0$
- $\exists\ c \in C\left(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}\right)$ the set of bounded and continuous functions on $\mathbb{R}^d \times \mathbb{R}$ taking values in $\mathbb{R}$ such that : $f(x,y) = c(x,y).y$

with

- $c(x,y) > 0, \forall x \in \mathbb{R}^d,\ y \in [0,1)$
- $c(x,y) \leq 0,\ \forall x \in \mathbb{R}^d, y > 1 \cup \mathbb{R}_-^*$
- $c(x) = \max_{y \geq 0} c(x,y) > 0$

and
$g \in C\left(\mathbb{R}^d, \mathbb{R}^+\right)$ the set of bounded and continuous functions on $\mathbb{R}^d$ taking values in $[0\,;+\infty\,[$ such that : $\sup_{x \in \mathbb{R}^d} g(x) = \bar{g} < \infty$.
Take $G_0 = \left\{x \in \mathbb{R}^d :\ g(x) > 0\right\}$, since g is continuous one notes $\overline{\mathring{G}_0} = \overline{G_0}$.

Our purpose is to combine effects of homogenization and large deviation principle regarding PDE parabolic case. This approach of combinatorial effects started with P. Baldi's work [1] which has been expended by Freidlin and Sowers [8] in stochastic differential equations (SDE). Thereafter, Diédhiou and Manga [4] proposed a survey of parabolic PDE in which homogenization prevails on the large deviation principle along with the condition of non-degeneracy. Recently Diédhiou [5], by a hypo-ellipticity condition of Hörmander, studies the case when the second order matrix of the operator $L_{\varepsilon,\delta_\varepsilon}$ is degenerate.
In this work, we analyse the situation in which homogenization parameter and large deviation parameter go at the same rate.
The difficulty on this case, is to give an explicit limit of the solution $u^{\epsilon,\delta}(t,x)$ as in [4] or [5] . We give here the superior and inferior limits of $\epsilon \log u^{\epsilon,\delta}(t,x)$, from which we can deduce the results given by [4].
Assumption and definition: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a d-dimensional Brownian motion $\left(W^1, \ldots, W^d\right)$ is defined. Let $\mathbb{E}$ the corresponding expectation operator. We have already defined $\langle .,. \rangle$ as the standard euclidean inner product on $\mathbb{R}^d$; let $\|\ .\ \|$ be the associated norm. Also let $\mathbb{T}^d$ be the d-dimensional torus of size 1 and $C^\infty\left(E; F\right)$ be the space of absolute

continuous mapping from E to F; let $\| \, . \, \|_{C^{\infty}(E;F)}$ be the associated supremum norm. Also we define $\mathcal{P}\left(\mathbb{T}^d\right)$ as the collection of all probability measure on $\mathbb{T}^d$.

The paper is organized as follow, in the first section we give some results of the LDP and in the second section we study the convergence of the solution $u^{\epsilon,\delta}$.

Consider the Markov diffusion process $X_t^{x,\varepsilon,\delta} \in \mathbb{R}^d$ governed by the operator :

$$\boldsymbol{L_{\varepsilon,\delta}} = \tfrac{\varepsilon}{2} \sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij} \left(\frac{x}{\delta}\right) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B^{\varepsilon,\delta} \left(\frac{x}{\delta}\right) \frac{\partial}{\partial x_i} \tag{2}$$

Where $(*)$ is the symbol of transposition. The trajectories of this process can be constructed with the help of the SDE:

$$\begin{cases} dX_t^{x,\varepsilon,\delta} = \sqrt{\varepsilon}\sigma\left(\frac{X_t^{x,\varepsilon,\delta}}{\delta}\right) dW_t + B^{\varepsilon,\delta}\left(\frac{X_t^{x,\varepsilon,\delta}}{\delta}\right) dt \\ X_0^{x,\varepsilon,\delta} = x \end{cases} \tag{3}$$

where $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ and $B^{\varepsilon,\delta} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are regular applications and 1-periodic in each coordinate of the argument.

The vector-valued function $B^{\varepsilon,\delta}$ is given by : $B^{\varepsilon,\delta} = \frac{\varepsilon}{\delta} B_0 + B_1$.
B_0, B_1 are $C^{\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ $\forall \varepsilon, \delta > 0$.

Since the two parameters δ (homogenization) and ε (large deviation) tend to zero, we consider a new defined parameter $\delta_\varepsilon = \delta$. We suppose that $\lim\limits_{\varepsilon\downarrow 0} \frac{\delta_\varepsilon}{\varepsilon} = \gamma$, where $\gamma > 0$ a constant . So, the homogenization parameter and the large deviation parameter go at the same rate.

Define $\tilde{X}_t^{x,\varepsilon,\delta_\varepsilon}$ by : $\boldsymbol{\tilde{X}_t^{x,\varepsilon,\delta_\varepsilon}} = \frac{1}{\delta_\varepsilon} X^{x,\varepsilon,\delta_\varepsilon}_{\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2 t}$

Then

$$\begin{cases} d\tilde{X}_t^{x,\varepsilon,\delta} = \sigma\left(\tilde{X}_t^{x,\varepsilon,\delta}\right) d\tilde{W}_t + \frac{\delta_\varepsilon}{\varepsilon} B^{\varepsilon,\delta}\left(\tilde{X}_t^{x,\varepsilon,\delta}\right) dt \\ \tilde{X}_0^{x,\varepsilon,\delta} = \frac{x}{\delta_\varepsilon} \end{cases} \tag{4}$$

where $\tilde{W}_t^{\varepsilon,\delta_\varepsilon} = \frac{\sqrt{\varepsilon}}{\delta_\varepsilon} W^{\varepsilon,\delta_\varepsilon}_{\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2 t}$ is a Brownian motion.

Let us consider the process $\left\{\tilde{X}_t^{\varepsilon,\delta_\varepsilon} : t \geq 0\right\}$ $\mathbb{T}^d$-valued which generator is defined by :

$$\tilde{L}_{\varepsilon,\delta_\varepsilon} = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \frac{\delta_\varepsilon}{\varepsilon}\sum_{i=1}^{d} B^{\varepsilon,\delta_\varepsilon}(x)\frac{\partial}{\partial x_i} \tag{5}$$

The process $\left\{\tilde{X}_t^{\varepsilon,\delta_\varepsilon} : t \geq 0\right\}$ $\mathbb{T}^d$-valued has a unique invariant measure m_γ^ϵ, and we have $m_\gamma^\epsilon \Rightarrow m_\gamma$, when $\epsilon \to 0$.
We suppose that the matrix $a = \sigma\sigma^*$ is strongly elliptic, the generator $\tilde{L}_{\varepsilon,\delta_\varepsilon}$ converges to the operator

$$\begin{aligned} L_\gamma &= \frac{1}{2}\sum_{i,j=1}^{d}(a)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_0^i(x)\frac{\partial}{\partial x_i} + \gamma\sum_{i=1}^{d} B_1^i(x)\frac{\partial}{\partial x_i} \\ &= L_0 + \gamma\sum_{i=1}^{d} B_1^i(x)\frac{\partial}{\partial x_i} \end{aligned} \tag{6}$$

1 Large deviation principle

Let us define, for each $T > 0$ and $x \in \mathbb{R}^d$,

$$\boldsymbol{g^\varepsilon_{T,x}(\theta)} = \varepsilon \log \mathbb{E}\left[\exp\left(\frac{1}{\varepsilon}\left\langle \theta, X_T^{x,\varepsilon,\delta_\varepsilon}\right\rangle\right)\right] \quad \varepsilon > 0,\ \theta \in \mathbb{R}^d$$

Also define the $\hat{X}_t^{x,\varepsilon,\delta_\varepsilon}$ the pullback on $\mathbb{T}^d$ of $\frac{1}{\delta_\varepsilon} X^{x,\varepsilon,\delta_\varepsilon}_{\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 t}$ under the natural projection on $\mathbb{R}^d$ to $\mathbb{T}^d$, as the solution of the $\mathbb{T}^d$-valued stochastic differential equation

$$\begin{cases} d\hat{X}_t^{x,\varepsilon,\delta_\varepsilon} &= \displaystyle\sum_{k=1}^{d} \hat{\sigma}_k\left(\hat{X}_t^{x,\varepsilon,\delta_\varepsilon}\right) \circ dW_t^k + \hat{\sigma}_0^{\varepsilon,\delta_\varepsilon}\left(\hat{X}_t^{x,\varepsilon,\delta_\varepsilon}\right) dt \\ \hat{X}_0^{x,\varepsilon,\delta_\varepsilon} &= x/\delta_\varepsilon \end{cases} \tag{7}$$

If we set

$$\begin{aligned} g_{t,x}^\varepsilon(\theta) =& \langle x,\theta\rangle + \varepsilon \log \mathbb{E}\left[\exp\left(\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2 \int_0^{\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 t} \left\langle B^{\varepsilon,\delta_\varepsilon}\left(\hat{X}_s^{x,\varepsilon,\delta_\varepsilon}\right), \theta\right\rangle ds\right)\right. \\ & \left. \times \exp\left(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 t} \sum_{k=1}^{d}\left\langle \sigma_k\left(\hat{X}_s^{x,\varepsilon,\delta_\varepsilon}\right), \theta\right\rangle dW_s^k\right)\right] \end{aligned} \tag{8}$$

We know that (see Freidlin and Sowers [8])

$$\lim_{\epsilon,\delta\downarrow 0} g_{t,x}^{\varepsilon,\delta}(\theta) = \langle x,\theta\rangle + t\overline{\mathcal{J}_2}(\theta) = g_{t,x}(\theta),$$

where

$$\begin{aligned} \overline{\mathcal{J}_2}(\theta) = \inf_{\{\phi\in C^\infty(\mathbb{T}^d)\}} \sup_{\{\mu\in\mathcal{P}(\mathbb{T}^d)\}} \int_{\mathbb{T}^d} &\left(\frac{1}{2}\sum_{k=1}^{d}\left(\langle\sigma_k(z),\theta\rangle - \hat{\sigma}_k\phi(z)\right)^2 + \langle B_1(z),\theta\rangle\right. \\ &\left. - \langle B_1(z), \nabla\phi(z)\rangle + \frac{1}{\gamma}\left(\langle B_0,\theta\rangle - L_0\phi(z)\right)\right)\mu(dz) \end{aligned} \tag{9}$$

Then by the theorem of Dembo A. et al.[3], the random variables $\{X_T^{x,\epsilon,\delta}, \epsilon > 0\}$, satisfy a large deviation principle with rate function $I_{T,x}$ defined by

$$I_{T,x}(z) = \sup_{\theta \in \mathbb{R}^d} \{\langle \theta, z \rangle - g_{T,x}(\theta)\}, \; z \in \mathbb{R}^d.$$

Thereafter let us define $\overline{\boldsymbol{C}}$ as :

$$\overline{\boldsymbol{C}} = \int_{\mathbb{T}^d} c(z) m_{_\gamma}(dz)$$

From Varadhan [3] we have

Theorem 1.1. *Let c be an element of $C^\infty\left(\mathbb{T}^d\right)$, $m_{_\gamma}$ be the invariant measure of L_γ and D a Borel subset of $C\left([0,t], \mathbb{R}^d\right)$. Define for $z \in \mathbb{T}^d$:*

$$\boldsymbol{\alpha(\gamma)} := \frac{\gamma^2}{2} \inf_{\phi \in C^\infty\left(\mathbb{T}^d\right)} \left\{ \int_{\mathbb{T}^d} \|\sigma(z)\nabla\phi(z)\|^2 \, m_{_\gamma}(dz) \right\} \tag{10}$$

$$\boldsymbol{\beta(\gamma)} := \frac{\gamma^2}{2} \inf_{\phi \in C^\infty\left(\mathbb{T}^d\right)} \sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} \left\{ \int_{\mathbb{T}^d} \|\sigma(z)\nabla\phi(z)\|^2 \, \mu(dz) \right\}. \tag{11}$$

Then

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{E}\left[1\!\!1_D\left(X_t^{x,\varepsilon,\delta_\varepsilon}\right) \exp\left\{ \frac{1}{\varepsilon} \int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) ds \right\} \right] \geq t\overline{C} - \inf_{\phi \in \mathring{D}} S_{0,t}^2(\phi) + t\alpha(\gamma)$$

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{E}\left[1\!\!1_D\left(X_t^{x,\varepsilon,\delta_\varepsilon}\right) \exp\left\{ \frac{1}{\varepsilon} \int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) ds \right\} \right] \leq t\overline{C} - \inf_{\phi \in \overline{D}} S_{0,t}^2(\phi) + t\beta(\gamma)$$

Proof.
Without loss of generality we assume that $\overline{C} = 0$.
Since δ_ϵ tends much more quickly to zero than ϵ, then the homogenization dominates. So we will treat ϵ as fixed for the homogenization and afterwards consider large deviations for the homogenized equation.
We will mainly use the fact that the mappings σ, C, B_1 are smooth and periodic with period one in each direction, therefore there are bounded and their differentials in the $d-$ dimensional torus $\mathbb{T}^d$, which is compact hence bounded in $\mathbb{R}^d$. There exist constants such that

$$k_0 \leq \hat{C}(x) \leq k_1, \; k_1' \leq \langle \nabla \hat{C}, B_1 \rangle(x) \leq k_2'.$$

Let $\hat{C}$ be the solution of the Poisson equation $L_\gamma \hat{C} = -C$. And let us set $\overline{X}_t^{x,\epsilon,\delta_\epsilon} = \frac{1}{\delta_\epsilon} X_t^{x,\epsilon,\delta_\epsilon}$, we have

$$\Big(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\Big)^2 \Big(\hat{C}(\overline{X}_t^{x,\epsilon,\delta_\epsilon}) - \hat{C}(\frac{x}{\delta_\epsilon})\Big) = \big(\frac{\delta_\epsilon}{\epsilon} - \gamma\big) \int_0^t \langle \nabla \hat{C}, B_1 \rangle (\overline{X}_s^{x,\epsilon,\delta_\epsilon}) ds + \frac{\delta_\epsilon}{\sqrt{\epsilon}} \int_0^t \langle \sigma \nabla \hat{C}(\overline{X}_s^{x,\epsilon,\delta_\epsilon}), dW_s \rangle - \int_0^t C(\overline{X}_s^{x,\epsilon,\delta_\epsilon}) ds,$$

So we have

$$\mathbb{E}\Big[1\!\!1_D(X) \exp\Big\{\frac{1}{\epsilon}\int_0^t C(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}) ds\Big\}\Big] = \int_D \exp\Big\{ -\big(\frac{\delta_\epsilon}{\epsilon}\big)^2 \big(\hat{C}(\overline{X}_t^{x,\epsilon,\delta_\epsilon}) - \hat{C}(\frac{x}{\delta_\epsilon})\big) + \frac{1}{\epsilon}\big(\frac{\delta_\epsilon}{\epsilon} - \gamma\big) \int_0^t \langle \nabla \hat{C}, B_1 \rangle (\overline{X}_s^{x,\epsilon,\delta_\epsilon}) ds + \frac{\delta_\epsilon}{\epsilon^{\frac{3}{2}}} \int_0^t \langle \sigma \nabla \hat{C}(\overline{X}_s^{x,\epsilon,\delta_\epsilon}), dW_s \rangle \Big\} d\mathbb{P}.$$

Let us set $A(s) = \sigma \nabla \hat{C}(\overline{X}_s^{x,\epsilon,\delta_\epsilon})$. By some estimations and using the Girsanov formula, we have

$$\begin{aligned}\mathbb{E}\Big[1\!\!1_D(X) \exp\Big\{\frac{1}{\epsilon}\int_0^t C(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}) ds\Big\}\Big] &= \int_D \exp\Big\{ -\big(\frac{\delta_\epsilon}{\epsilon}\big)^2 \big(\hat{C}(\overline{X}_t^{x,\epsilon,\delta_\epsilon}) - \hat{C}(\frac{x}{\delta_\epsilon})\big) \\ &\quad + \frac{1}{\epsilon}\big(\frac{\delta_\epsilon}{\epsilon^2} - \gamma\big) \int_0^t \langle \nabla \hat{C}, B_1 \rangle (\overline{X}_s^{x,\epsilon,\delta_\epsilon}) ds + \frac{1}{2}\frac{\delta_\epsilon^2}{\epsilon^3} \int_0^t ||A_s||^2 ds \Big\} d\tilde{\mathbb{P}}_{X^{x,\epsilon,\delta_\epsilon}} \\ &\leq \exp\Big\{ K_1 \big(\frac{\delta_\epsilon}{\epsilon}\big)^2 + K_2 \frac{1}{\epsilon}\big(\frac{\delta_\epsilon}{\epsilon} - \gamma\big) t + \big(\frac{\delta_\epsilon}{\epsilon^{\frac{3}{2}}}\big)^2 \frac{\beta(\gamma)}{\gamma^2} t \Big\} \times \tilde{\mathbb{P}}_{X^{x,\epsilon,\delta_\epsilon}}(D)\end{aligned}$$

where

$$\frac{d\tilde{\mathbb{P}}_{X^{x,\epsilon,\delta_\epsilon}}}{d\mathbb{P}_{X^{x,\epsilon,\delta_\epsilon}}} = \exp\Big\{ \frac{\delta_\epsilon}{\epsilon^{\frac{3}{2}}} \int_0^t \langle A(s), dW_s \rangle - \frac{1}{2}\big(\frac{\delta_\epsilon}{\epsilon^{\frac{3}{2}}}\big)^2 \int_0^t \| A(s) \|^2 \, ds \Big\}.$$

Since $X^{x,\epsilon,\delta_\epsilon}$ is solution of the SDE

$$X_t^{x,\epsilon,\delta_\epsilon} = x + \int_0^t f^\epsilon(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}) ds + \sqrt{\epsilon} \int_0^t \sigma(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}) d\hat{W}_s,$$

where $\hat{W}_t = W_t - \frac{\delta_\epsilon}{\epsilon^{\frac{3}{2}}} \int_0^t \sigma \nabla \hat{C}(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}) ds$, $f^\epsilon = B^{\epsilon,\delta_\epsilon} + \frac{\delta_\epsilon}{\epsilon} \sigma \cdot (\sigma \nabla \overline{C})$. The rate function under $\tilde{\mathbb{P}}$ is S_{0t}.

Then

$$\lim_{\epsilon \downarrow 0} \epsilon \log \mathbb{E}\Big[1\!\!1_D(X) \exp\Big\{\frac{1}{\epsilon}\int_0^t C(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}) ds\Big\}\Big] \leq -\inf_{\varphi \in \overline{D}} S_{0t}(\varphi) + t\beta(\gamma).$$

For the other inequality we can notice that

$$\mathbb{E}\Big[1\!\!1_D(X)\exp\Big\{\frac{1}{\epsilon}\int_0^t C(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon})ds\Big\}\Big] = \int_D \exp\Big\{-(\frac{\delta_\epsilon}{\epsilon})^2\Big(\hat{C}(\overline{X}^{x,\epsilon,\delta_\epsilon}_{\frac{\epsilon}{\delta_\epsilon^2}t}) - \hat{C}(\frac{x}{\delta_\epsilon})\Big)$$

$$+\frac{1}{\epsilon}(\frac{\delta_\epsilon}{\epsilon}-\gamma)\int_0^t\langle\nabla\hat{C},B_1\rangle(\overline{X}_s^{x,\epsilon,\delta_\epsilon})ds + \frac{\delta_\epsilon}{\epsilon^{\frac{3}{2}}}\int_0^t\langle\sigma\nabla\hat{C}(\overline{X}_s^{x,\epsilon,\delta_\epsilon}),dW_s\rangle\Big\}d\mathbb{P}$$

$$\geq \exp\Big\{(\frac{\delta_\epsilon}{\epsilon})^2K_1' + \frac{1}{\epsilon}(\frac{\delta_\epsilon}{\epsilon}-\gamma)K_2't\Big\}\times\int_D\exp\Big\{\frac{1}{2}(\frac{\delta_\epsilon}{\epsilon^{\frac{3}{2}}})^2\int_0^t \|\ A(s)\ \|^2\ ds\Big\}d\tilde{\mathbb{P}}$$

$$\geq \exp\Big\{(\frac{\delta_\epsilon}{\epsilon})^2K_1' + \frac{1}{\epsilon}(\frac{\delta_\epsilon}{\epsilon}-\gamma)K_2't + \frac{\alpha(\gamma)}{\gamma^2}t\Big\}\tilde{\mathbb{P}}(D).$$

And we end the proof by showing that the limit of

$$\epsilon\log\mathbb{E}\Big[1\!\!1_D(X)\exp\Big\{\frac{1}{\epsilon}\int_0^t C(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon})ds\Big\}\Big]$$

is greater than $-\min_{\varphi\in\mathring{D}} S_{0t}(\varphi)+\alpha(\gamma)t.$ □

2 Convergence of $u^{\varepsilon,\delta_\varepsilon}$

In order to analyse the behaviour of $u^{\varepsilon,\delta_\varepsilon}$ use the representative of Feynman-kac

$$\boldsymbol{u^{\varepsilon,\delta_\varepsilon}(t,x)} = \mathbb{E}\left[g\left(X_t^{x,\varepsilon,\delta_\varepsilon}\right)e^{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},Y_s^{x,\varepsilon,\delta_\varepsilon}\right)ds}\right] \tag{12}$$

Where $Y^{x,\varepsilon,\delta_\varepsilon}$ is the progressive measurable solution associated of the BSDE introduced by Pardoux and Peng [9] :

$$\begin{cases} Y_t^{x,\varepsilon,\delta_\varepsilon} = g\left(X_t^{x,\varepsilon,\delta_\varepsilon}\right) + \frac{1}{\varepsilon}\int_s^t f\left(\frac{X_r^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},Y_r^{x,\varepsilon,\delta_\varepsilon}\right)dr - \frac{1}{\sqrt{\varepsilon}}\int_s^t Z_r^{x,\varepsilon,\delta_\varepsilon}dW_r \\ \mathbb{E}\left\{\int_s^t \left|Z_r^{x,\varepsilon,\delta_\varepsilon}\right|^2 dr\right\} < \infty \end{cases} ,0\leq s\leq t$$

Since $Y_0^{x,\varepsilon,\delta_\varepsilon} = u^{\varepsilon,\delta_\varepsilon}(t,x)$ and $0 < Y_0^{x,\varepsilon,\delta_\varepsilon} < 1$. Let's introduce

$$\boldsymbol{v^{\varepsilon,\delta_\varepsilon}(t,x)} = \varepsilon\log u^{\varepsilon,\delta_\varepsilon}(t,x),$$

as in Pradeilles [10], we observe that $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is a viscosity solution of :

$$\begin{cases} \frac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) = L_{\varepsilon,\delta_\varepsilon}v^{\varepsilon,\delta_\varepsilon}(t,x) + \frac{1}{2}\left\|\nabla v^{\varepsilon,\delta_\varepsilon}(t,x)\sigma\left(\frac{x}{\delta_\varepsilon}\right)\right\|^2 + c\left(\frac{x}{\delta_\varepsilon},u^{\varepsilon,\delta_\varepsilon}(t,x)\right) \\ v^{\varepsilon,\delta_\varepsilon}(0,x) = \varepsilon\log\left(g(x)\right), \qquad x\in G_0 \\ \lim_{t\to 0}v^{\varepsilon,\delta_\varepsilon}(t,x) = -\infty, \qquad x\in\mathbb{R}^d\backslash G_0 \end{cases} \tag{13}$$

Remark 2.1. If $\lim_{\epsilon\downarrow 0} \epsilon \log u^{\epsilon,\delta}(t,x) < 0$, then for ϵ small enough, there exists constant $C > 0$ such that $u^{\epsilon,\delta}(t,x) < e^{-\frac{C}{\epsilon}}$, and since $u^{\epsilon,\delta}(t,x) = Y_0^{\epsilon,\delta,x}$.
We have $Y_0^{\epsilon,\delta,x} \geq \mathbb{E}(g(X_t^{\epsilon,\delta,x})$, and then

$$\lim_{\epsilon\downarrow 0} \epsilon log \mathbb{P}(X_t^{\epsilon,\delta,x} \in G_0) < 0.$$

Definition 2.2. A functional $\tau : \mathcal{C}([0,t],\mathbb{R}^d) \to [0,t]$ is a stopping time if for all $\varphi, \tilde{\varphi} \in \mathcal{C}([0,t],\mathbb{R}^d)$ and $s \in [0,t], \varphi_r = \tilde{\varphi}_r$, for all $r \in [0,s]$ and $\tau(\varphi) < s$ imply $\tau(\varphi) = \tau(\tilde{\varphi})$.
Let us set Σ_t the set of stopping times, and Θ_t the set elements of Σ_t such that there exists $\mathcal{O}$ such that for all

$$\varphi \in \mathcal{C}\Big([0,\infty[,\mathbb{R}^d\Big); \quad \tau(\varphi) = \inf\Big\{s \leq t : (t-s,\varphi_s) \in \mathcal{C}([0,\infty[,\mathbb{R}^d)\Big\},$$

with the convention $\inf \emptyset = t$.
τ is hence a well defined element of Σ_t, and $\mathcal{O}$ is an open set associated.

$$\tau : [0,\infty[\times \mathcal{C}([0,\infty[,\mathbb{R}^d) \to [0,+\infty[$$

is an element of Σ(resp. Θ) if and only if, for all $t \geq 0, \tau_t = \tau(t,.) \in \Sigma_t$ (resp.Θ_t) where $\tau(t,\varphi) = \tau_t(\varphi_{[0,t]})$.

Let us set $\mathcal{E} = \Big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d;\ \overline{\lim_{\epsilon\downarrow 0}}\, \epsilon \log u^{\epsilon,\delta}(t,x) < 0\Big\}$.

Proposition 2.3. *We have* $\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t,x) = 0$, *uniformly in all compact set* $\mathcal{K}$ *of* $\mathcal{E}$.

Proof: We know $\overline{\lim_{\epsilon\downarrow 0}}\, \epsilon \log u^{\epsilon,\delta}(t,x)$ is continuous, since $u^{\epsilon,\delta}(t,x)$ is continuous. Let $\mathcal{K}$ be a compact set of $\mathcal{E}$ by the remark above, there exists $C > 0$, such that for all $\forall (t,x) \in \mathcal{K}$,

$$\overline{\lim_{\epsilon\downarrow 0}}\ \epsilon \log u^{\epsilon,\delta}(t,x) < -C.$$

Hence, there exists $\epsilon_{\mathcal{K}}$ such that

$$\forall 0 < \epsilon < \epsilon_{\mathcal{K}},\ \forall (t,x) \in \mathcal{K},\ u^{\epsilon,\delta}(t,x) < e^{-\frac{C}{\epsilon}}.$$

Let us set

$$V_1(t,x) = \inf_{\tau\in\Theta_t} \sup_{\varphi} \Big\{\big(\overline{C} + \alpha(\gamma)\big)\tau - S_{0\tau}(\varphi),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \varphi \in \mathcal{C}([0,t])\Big\},$$

$$V_2(t,x) = \inf_{\tau\in\Theta_t} \sup_{\varphi} \Big\{ \big(\overline{C} + \beta(\gamma)\big)\tau - S_{0\tau}(\varphi),\ \varphi_0 = x, \varphi_t \in G_0,\ \varphi \in \mathcal{C}([0,t]) \Big\}.$$

We can see that

$$V_1(t,x) \le V_2(t,x) \le 0.$$

Proposition 2.4. *We have*

$$V_1(t,x) \le \varliminf_{\epsilon\downarrow 0} \epsilon \log u^{\epsilon,\delta}(t,x) \le \varlimsup_{\epsilon\downarrow 0} \epsilon \log u^{\epsilon,\delta}(t,x) \le V_2(t,x).$$

Proof: We have,

$$\begin{aligned} Y_0^{\epsilon,\delta,x} &= \mathbb{E}\Big[Y_\tau^{\epsilon,\delta,x} \exp(\frac{1}{\epsilon}\int_0^t C(\frac{X_r}{\delta}, Y_r^{\epsilon,\delta,x})dr)\Big] \\ &\le \mathbb{E}[Y_\tau^{\epsilon,\delta,x} \exp(\frac{1}{\epsilon}\int_0^t C(\frac{X_r}{\delta})dr)] \\ &\le (1\vee \overline{g})\mathbb{E}[1\!\!1_{\{\tau<t\}} \exp(\frac{1}{\epsilon}\int_0^t C(\frac{X_r^{\epsilon,\delta}}{\delta})dr)] + \mathbb{E}[1\!\!1_{\{X_t^{\epsilon,\delta}\in G_0\}} 1\!\!1_{\{\tau=t\}} \exp(\frac{1}{\epsilon}\int_0^t C(\frac{X_r^{\epsilon,\delta}}{\delta})dr)] \\ &\le (1\vee \overline{g})\mathbb{E}[1\!\!1_{\{\tau<t\}\cup\{X_t^{\epsilon,\delta}\in G_0,\tau=t\}} \exp(\frac{1}{\epsilon}\int_0^t C(\frac{X_r^{\epsilon,\delta}}{\delta})dr)] \end{aligned} \tag{14}$$

using the theorem (1.1), we get

$$\varlimsup_{\epsilon\downarrow 0} \epsilon \log u^{\epsilon,\delta}(t,x) \le \inf_{\tau\in\Theta_t} \{(\overline{C}+\beta(\gamma))\tau - \inf_{\varphi} S_{0\tau}(\varphi), \varphi_0 = x, \varphi_t \in G_0, \varphi \in \mathcal{C}([0,t])\}.$$

For the other inequality, we set

$$\tau(\varphi) = \inf_{\varphi}\{0 \le s \le t : \varliminf_{\epsilon\downarrow 0} u^{\epsilon,x,\delta}(t-s,\varphi_s) > -\alpha\}, \alpha > 0,$$

then we have $\tau(\varphi) < t$. Let choose φ such that

$$(\overline{C} + \beta(\gamma))\tau - \inf_{\varphi} S_{0\tau}(\varphi) > V_1(t,x) - \alpha.$$

Let $\beta \in]0, t-\tau(\varphi)[$ such that for all $s \in [\tau(\varphi), \tau(\varphi)+\beta]$,

$$-\frac{5\alpha}{4} \le \varliminf_{\epsilon\downarrow 0} u^{\epsilon,x,\delta}(t,x) \le -\frac{3\alpha}{4}.$$

τ is supper semi-continuous, then there exists $\eta > 0$ such that

$$||\psi - \varphi||_\beta \equiv \sup_{s\in[0,t]} |\psi_t - \varphi_s| < \eta \Longrightarrow \tau(\psi) < \tau(\varphi) + \beta.$$

Let us set $u^*(t,x) = \underline{\lim} u^{\epsilon,x,\delta}(t,x)$. Choosing $\eta > 0$ such that

$$||\psi - \varphi||_\beta \leq \eta \Rightarrow \forall s \in [0, \tau(\varphi) + \beta],\ |u^*(t-s,\psi_s) - u^*(t-s,\varphi_s)| \leq \frac{\alpha}{4},$$

we have

$$\begin{aligned}
Y_0^{\epsilon,x,\delta} &= \mathbb{E}[Y^{\epsilon,x,\delta}_{\tau(\varphi)+\beta} \exp(\frac{1}{\epsilon}\int_0^{\tau(\varphi)+\beta} C(\frac{X_r}{\delta}, Y_r^{\epsilon,\delta,x})dr)] \\
&= \mathbb{E}[Y^{\epsilon,x,\delta}_{\tau(\varphi)+\beta} 1\!\!1_{\{||X-\varphi||_\beta < \eta\}} \exp(\frac{1}{\epsilon}\int_0^{\tau(\varphi)+\beta} C(\frac{X_r}{\delta}, Y_r^{\epsilon,\delta,x})dr)] \\
&+ \mathbb{E}[Y^{\epsilon,x,\delta}_{\tau(\varphi)+\beta} 1\!\!1_{\{||X-\varphi||_\beta \geq \eta\}} \exp(\frac{1}{\epsilon}\int_0^{\tau(\varphi)+\beta} C(\frac{X_r}{\delta}, Y_r^{\epsilon,\delta,x})dr)] \\
&\geq \mathbb{E}[Y^{\epsilon,x,\delta}_{\tau(\varphi)+\beta} 1\!\!1_{\{||X-\varphi||_\beta < \eta\}} \exp(\frac{1}{\epsilon}\int_0^{\tau(\varphi)+\beta} C(\frac{X_r}{\delta}, Y_r^{\epsilon,\delta,x})dr)]
\end{aligned} \tag{15}$$

Let us choose $\gamma > 0$, for $\epsilon > 0$ small enough the proposition (2.4)holds, and using the fact that the function C is continuous, we get

$$Y_0^{\epsilon,x,\delta} \geq \mathbb{E}\Big[Y^{\epsilon,x,\delta}_{\tau(\varphi)+\beta} 1\!\!1_{\{||X-\varphi||_\beta < \eta\}} \exp(\frac{1}{\epsilon}\int_0^{\tau(\varphi)+\beta} C(\frac{X_r}{\delta})dr - 2\alpha - \gamma)\Big]$$

so, we deduce that

$$u^*(t,x) \geq \sup_\psi \{(\overline{C}+\alpha(\gamma))(\tau(\varphi)+\beta) - S_{0\tau(\varphi)+\beta}(\psi), \psi = x,\ ||X-\varphi||_\beta < \eta\} - 2\alpha,$$

hence

$$\forall \alpha, \beta > 0,\ u^*(t,x) \geq (\overline{C}+\alpha(\gamma))(\tau(\varphi)+\beta) - \inf_\psi S_{0\tau(\varphi)+\beta}(\psi) - 2\alpha \geq V_1(t,x) - 3\alpha,$$

and then $u^*(t,x) \geq V_1(t,x)$.

Remark 2.5. We have

$$\lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x) = 1,$$

in all compact subset of the set $\{(t,x) \in [0,+\infty[\times \mathbb{R}^d : V_1(t,x) = 0\}$, and

$$\lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x) = 0,$$

in all compact subset of the set $\{(t,x) \in [0,+\infty[\times \mathbb{R}^d : V_2(t,x) < 0\}$.

References

[1] Baldi P., *Large deviation for diffusion processes with homogenization and applications*, Ann. Probab. **19**, 509-524, 1991. http://dx.doi.org/10.1214/aop/1176990438

[2] Baxendale P. H., Stoock D. W., *Large deviations and stochastic flows of diffeomorphisms*, Proba. TH. Rel. Fields, **80**, 169-215, 1988. http://dx.doi.org/10.1007/bf00356102

[3] Dembo A., Zeitouni O., *Large deviation Techniques and Applications*, Jones and Bartlet Publishers, Boston, 1993.

[4] Diédhiou A., Manga C., *Application of homogenization and large deviation to a parabolic semi-linear equation*, J. Math. Anal. Appl. **342**, 146-160, 2008. http://dx.doi.org/10.1016/j.jmaa.2007.11.037

[5] Diédhiou A., *Limit of the solution of a PDE in the degenerate case*, App. Math. **4** , 338-342, 2013 (http://www.scirp.org/journal/am)

[6] Deuschel J. D., Stroock D. W. *A function space large deviation principle for certain stochastic integral*, Proba. Th. Rel. Fields, **83**, 279-307, 1989. http://dx.doi.org/10.1007/bf00333151

[7] Evans Lawrence C., *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. , **111**, 359-375 , 1989. http://dx.doi.org/10.1017/s0308210500018631

[8] Freidlin M.I., Sowers R.B., *A comparison of homogenization and large deviation, with applications to wave-fronts propagation*, Stochastic Process. Appl. **82**, 23-52, 1999. http://dx.doi.org/10.1016/s0304-4149(99)00003-4

[9] Pardoux E., Peng S *Backward stochastic differential equations and quasilinear parabolic differential equations*, Lecture Notes in Control. and Inform. Sci. **176**, 200-217, 1992. http://dx.doi.org/10.1007/bfb0007334

[10] Pradeilles F., *Wave-fronts propagations for reaction-diffusion systems and backward SDEs*, The Ann. of Proba. vol **26**, n°4, 1575-1613, 1998

[11] Stroock W. D., *On the rate at which a homogeneous diffusion approaches a limit, an application of large deviation theory to certain stochastic integrals*,The Ann. of Proba. vol **14**, n°3, 840-859, 1986. http://dx.doi.org/10.1214/aop/1176992441

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