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Exact Penalty Method for Minimizing of Nonsmooth Functions on Convex Sets

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Abstract

This paper considers a constrained nonsmooth optimization problem in which an objective function is locally Lipschitz and constraint functions are convex. With the help of exact penalty functions this problem is transformed into an unconstrained one. A regularity condition under which there exists an exact penalty parameters is introduced. For its implementation it is necessary that functions defining constraints were nonsmooth at every boundary point of this set. It is shown that in some cases it is possible to find an analytic representation of an exact penalty parameter.

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1 Introduction. Penalty Function Method

The method of penalty functions was introduced by Courant in 1943 [1] by the reason related to the physical nature of the problem under consideration. The basic idea of the method is to reduce the problem of minimizing with constraints to the problem of minimizing a function without any constraints. The auxiliary function is chosen in such a way that it coincides with the objective function to be minimized in the set and it quickly grows out of it. Penalty function methods require the solution of a sequence of (increasingly ill-conditioned) unconstrained optimization problems.

Consider the constrained optimization problem

$$f(x) \to \inf_{x \in X}, \tag{1}$$

where $f : R^n \to R$ is a locally Lipschitz function. Suppose that the set $X \subset R^n$ is nonempty, compact and given in the form $X = \{\ x \in R^n \mid \varphi(x) = 0\}$, where $\varphi : R^n \to R, \quad \varphi(x) \geq 0 \quad \forall x \in R^n$, is a non-negative locally Lipschitz function. Then the set X is a closed set of global minimum points of φ on R^n. To solve (1) by the method of penalty functions, we usually introduce the function $F(c,x) = f(x) + c\varphi(x)$, where c is a non-negative number, called the penalty parameter, and consider the unconstrained optimization problem

$$F(c,x) \to \inf_{x \in R^n}. \tag{2}$$

Assume that the infimum in (2) is attained for every $c > 0$. Let

$$f^* = \min_{x \in X} f(x), \quad F^*(c) = \min_{x \in R^n} F(c,x), \quad x(c) = \arg\min_{x \in R^n} F(x,c).$$

$$x^{**} = \arg\min_{x \in R^n} f(x), \quad f^{**} = f(x^{**}).$$

Suppose that $f^{**} > -\infty$. Note that $f^* \geq F^*(c)$ for any positive c. Choose a monotonically increasing sequence of non-negative numbers $\{c_k\}(k = 0, 1, 2, \ldots)$ $0 = c_0 < c_1 < c_2 < \ldots < c_k < \ldots, \quad c_k \to +\infty$. Then $x^{**} = x(c_0) = f^{**}$.

Theorem 1.[2]. If $x(c_k)$ is a sequence of solutions of auxiliary problems (2), then the following inequalities

1. $F^*(c_k) = F(c_k, x(c_k)) \leq F(c_{k+1}, x(c_{k+1})) = F^*(c_{k+1}) \quad \forall k > 0;$

2. $f(x(c_k)) \leq f(x(c_{k+1})); \quad f(x(c_k)) \leq F(c_k, x(c_k)) \leq f^*;$

3. $\varphi(x(c_{k+1})) \leq \varphi(x(c_k)) \quad \forall k > 0; \quad \varphi(x(c_k)) \to 0, \quad \text{if} \ \ c_k \to +\infty,$

hold. Denote

$$\mathcal{L}(\varphi, x^{**}) = \{x \in R^n \mid \varphi(x) \leq \varphi(x^{**})\}.$$

The set $\mathcal{L}(\varphi, x^{**})$ is compact. Thus, the sequence of points $\{x(c_k)\}$ obtained as a result of global minimization of the family of functions $F(c_k, x)$ is a minimizing sequence for the function φ.

2 Exact Penalty Functions

In the theory of penalty functions exact penalty functions are of particular interest. For exact penalty functions there exists a penalty parameter c^*, such that for every $c \geq c^*$ the set of minimum points of $F(c, x)$ on R^n coincides with the set of solutions of (1). This parameter c^* is called an exact penalty parameter for the family of functions $F(c, x)$. For the first time Erimin I. [3] and Zangwill W. [4] established a similar global result for the convex problem. There are a lot of papers in which exact penalty functions are investigated, for example in [5]-[16].

Consider the optimization problem:

$$f(x) \to \inf_{x \in X}, \quad X = \{x \in R^n \mid \ f_1(x) \leq 0\}, \tag{3}$$

where $f : R^n \to R$ is locally Lipschitz and $f_1 : R^n \to R$ is convex. Assume that X does not consist of an isolated point and there exists a point $\hat{x} \in X$ for which the inequality $f_1(\hat{x}) < 0$ holds. Slater's condition states that the set X has interior points. Let the global minimizers of f do not belong to X. Consider the optimization problem:

$$f(x) \to \inf_{x \in X}, \tag{4}$$

where

$$X = \{x \in R^n \mid \ \varphi(x) = 0\}, \quad \varphi(x) = \max\{0, f_1(x)\}. \tag{5}$$

Under our assumptions it is easy to see that problems (3) and (4) are equivalent. To solve (4) we use the method of exact penalty functions. Introduce the function

$$F(c, x) = f(x) + c\varphi(x), \quad c > 0,$$

and consider the unconstrained optimization problem

$$F(c, x) \to \inf_{x \in R^n}. \tag{6}$$

Assume that the infimum in (6) is attained for any positive number c. The boundedness of the Lebesgue set

$$\mathcal{L}(f, x_0) = \{x \in R^n \mid f(x) \leq f(x_0)\}, \quad x_0 \in R^n$$

is a sufficient condition for this.

In our case φ is convex and satisfies the following expansion

$$\varphi(x + \alpha g) = \varphi(x) + \alpha\varphi'(x, g) + o(\alpha, x, g), \ x, g \in R^n, \ \alpha \in R,$$

where $\varphi'(x, g)$ is a directional derivative of φ at x and

$$\frac{o(\alpha, x, g)}{\alpha} \to 0, \quad \text{if } \alpha \downarrow 0.$$

In this case, $f_1'(x,g) = \max\limits_{v\in\partial f_1(x)} \langle v, g\rangle$, $\varphi'(x,g) = \max\limits_{v\in\partial\varphi(x)} \langle v, g\rangle$, where $\partial f_1(x)$, $\partial\varphi(x)$ are subdifferentials of f_1 and φ at x. It is easy to see that if $x \in \text{int } X$, then $\partial\varphi(x) = 0_n$. If $x \in \text{bd } X$, then $\partial\varphi(x) = \text{co }\{0_n, \partial f_1(x)\}$, where bd A denotes the boundary of A and co$A$ denotes the convex hull of A.

Since minimizers of f_1 do not belong to X then the zero point is a boundary point of the subdifferential $\partial\varphi(x)$ at every boundary point $x \in X$.

Let $\Gamma(X,x)$ be a cone of feasible directions at $x \in X$

$$\Gamma(X,x) = \text{cl }\{g \in R^n \mid \exists\alpha_0 > 0, \quad x + \alpha g \in X \quad \forall\alpha \in [0, \alpha_0)\},$$

where cl A is a closure of A. In this case the cone $\Gamma(X,x)$ is convex and closed. A set

$$N(X,x) = \{g \in R^n \mid \langle g, z - x\rangle \leq 0 \ \forall z \in X\}$$

is called the normal cone to X at the point $x \in X$ [17]. The cone $N(X,x)$ is also closed and convex. Under the above assumptions at the point $x \in \text{bd } X$, $(f_1(x) = 0)$ the equalities

$$\Gamma(X,x) = -\left[\text{cone }\{\partial(f_1(x))\}\right]^*, \ N(X,x) = -[\text{cone }(X - x)]^* = -\Gamma^*(X,x)$$

hold, where $\Gamma^*(X,x)$ is a conjugate cone of $\Gamma(X,x)$.

3 Regularity Condition for the Existence of an Exact Penalty Parameter

Under considering optimization problems with constraints the properties of the functions f and φ play an important role for the existence of an exact penalty parameter.

Choose a point $z \in R^n$ which does not belong to the set X and project z onto X. Suppose that a point $x \in X$ is the projection of z ($x = \text{pr}(z)$) onto the set X in the Euclidean norm. Consider the set

$$\mathcal{A}(X) = \{x \in \text{bd}X \mid \exists z \notin X, \ x = \text{pr }(z)\}.$$

In this case the normal cone $N(X,x)$ is not only a zero vector at every point $x \in \mathcal{A}(X)$. If $z \notin X$ and $x = \text{pr}(z)$, then the inclusion

$$(z - x) \in N(X,x)$$

holds. Further we will assume that for the function φ a regularity condition takes place [2]:
for each boundary point $x^* \in \text{bd }(X)$ *there exist positive numbers* $\varepsilon(x^*) > 0$ *and* $\beta(x^*) > 0$*, such that the inequality*

$$\frac{o(\alpha, x, g)}{\alpha} > -\varphi'(x,g) + \beta(x^*) \tag{7}$$

holds, where

$$\forall \alpha \in (0, \varepsilon(x^*)],\ \forall x \in \mathcal{A}(X) \cap B(x^*, \varepsilon(x^*)),\ \forall g \in N(X, x), ||g|| = 1.$$

Here and below $B(x_0, r)$ is a closed ball of a radius r centered at x_0. This regularity condition is more constructive than the analogous condition in [7].

Theorem 2. If the set X is compact and for the function φ regularity condition (7) holds for the family of penalty functions then there exists an exact penalty parameter c^*, i.e.

$$x(c_k) \in X \quad \forall c_k \geq c^*.$$

Remark 1. For an exact penalty parameter to exist, it is necessary that the function φ was nondifferentiable at each boundary point of X.

Lemma 1. If for a function φ, which determines the set X of form (5), at a boundary point $x \in$ bd (X) regularity condition (7) holds, then

$$\varphi'(x^*, g) \geq \beta(x) \quad \forall g \in N(X, x), \quad ||g|| = 1.$$

Corollary 1. If for the function φ at the point $x \in$ bd (X) regularity condition (7) holds, then

$$\min_{\substack{||g|| = 1 \\ g \in N(X,x)}} \varphi'(x, g) \geq \beta(x).$$

Suppose that the function φ is convex and for it regularity condition (7) holds at $x \in$ bd X. Then

$$N(X, x) = \text{cl cone } (\partial\varphi(x)),$$

and

$$\min_{\substack{\|g\| = 1, \\ g \in N(X,x)}} \varphi'(x, g) = \beta(x) > 0, \qquad \inf_{x \in \text{ bd } X} \beta(x) = \beta(X) > 0.$$

Thus, the function φ can be used for constructing of a family of penalty functions.

3.1 Calculation of Exact Penalty Parameters

Show that if X is defined with using a strongly convex function f_1, then there always exists an exact penalty parameter. Let $f_1 : R^n \to R$ be a strongly convex function and $m > 0$ be its constant of strongly convexity From the convex analysis [17] it is known that

$$\langle v(x) - v(y), x - y \rangle \geq 2m\|x - y\|^2 \quad \forall v(x) \in \partial f_1(x),\ \forall v(y) \in \partial f_1(y).$$

Let x_1^* be a minimizer of f_1 on R^n. As the Slater condition for X is satisfied then $x_1^* \in \text{int } X$. Therefore

$$||v(x)|| \geq 2m||x - x_1^*|| \quad \forall x \in R^n, \quad \forall v(x) \in \partial f_1(x), \tag{8}$$

and

$$||v(x)||^2 \geq 4m(f_1(x) - f_1(x_1^*)) \quad \forall x \in R^n, \quad \forall v(x) \in \partial f_1(x).$$

Furthermore, for each $x_0 \in R^n,\ f(x_0) > f(x_1^*)$ the level set

$$L(f_1, x_0) = \{x \in R^n \mid \ f_1(x) \leq f_1(x_0)\}$$

is compact. Let d be the maximum radius of a closed ball $B(x_1^*, d)$ contained in X. Then, from (8) we have $||v(x)|| \geq 2mr \quad \forall v(x) \in \partial f_1(x), \quad \forall x \notin B(x_1^*, d)$. Thus, for every boundary point of X the inequality

$$||v(x)|| \geq 2mr \quad \forall x \in \text{bd } X$$

holds.

Theorem 3. If the function f_1 is strongly convex, $\varphi(x) = \max\{0, f_1(x)\}$, the set X is given in the form (5) and the Slater condition for the set X is satisfied, then regularity condition (7) is satisfied at every boundary point.

Corollary 2. The value of $\frac{L}{2md}$, where L is a Lipschitz constant of the objective function f on the level set $\mathcal{L}(\varphi, x^{**})$ can be taken as an exact penalty parameter.

Example 1. Let

$$f_1(x) = \langle Ax, x \rangle - 1, \quad \varphi(x) = \max\{0, f_1(x)\}, \quad X = \{x \in R^n \mid \varphi(x) = 0\},$$

where A is a positive definite matrix. The set X is an ellipsoid. If $x \in \text{bd } X$, then

$$N(X, x) = \{g \in R^n \mid \ g = 2\lambda Ax,\ \lambda \geq 0\}, \quad \partial\varphi(x) = \max\{0, 2Ax\}.$$

If $g = \dfrac{Ax}{||Ax||} \in N(X, x)$, then

$$\min_{\substack{\|g\| = 1, \\ g \in N(X,x)}} \varphi'(x, g) = 2||Ax||,$$

and $\min\limits_{x \in \text{bd } X} 2||Ax|| \geq 2\lambda_{min} d = \dfrac{2\lambda_{min}}{\lambda_{max}}$, where λ_{min} and λ_{max} are minimum and maximum eigenvalues of the matrix A.

4 Conclusions

The paper presents a method of exact penalty functions. The regularity condition imposed on the functions which define the set of constraints is considered. If this condition is satisfied then there is an exact penalty parameter. Note that the objective function is required to be only local Lipschitz and bounded below. In the case of a convex set the sufficient conditions are given which allow to find analytical representation of an exact penalty parameter. Examples of calculation of such parameters for different types of convex sets which defined with the help of strongly convex functions are described.

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References

[1] R. Courant, Variational methods for the solution of problems of equilibrium and vibrations, *Bulletin of the American Mathematical Society*, **49** (1943), 1-23. http://dx.doi.org/10.1090/s0002-9904-1943-07818-4

[2] L.N. Polyakova, On the method of exact penalty quasidifferentiable functions, (In Russian), *J. of Computational Mathematics and Mathematical Physics*, **41** (2001), 225-238.

[3] I.I. Eremin, Penalty method of the convex programming (In Russian), Reports of the USSR Academy of Sciences, **143** (1967), 748-751.

[4] W. Zangwill, Non-linear programming via penalty function, *Management Science*, **13** (1967), 344-358. http://dx.doi.org/10.1287/mnsc.13.5.344

[5] D.P. Bertsekas, Necessary and sufficient conditions for a penalty method to be exact, *Mathematical Programming*, **9** (1975), 87-99. http://dx.doi.org/10.1007/bf01681332

[6] A.R. Conn, Constrained optimization using a nondifferentiable penalty function, *SIAM Journal on Numerical Analysis*, **10** (1973), 760-784. http://dx.doi.org/10.1137/0710063

[7] V.F. Demyanov, Exact penalty functions in problems of nonsmooth optimization (In Russian), *Saint-Petersburg Academy of Sciences*, **4** (1994), 21-27.

[8] V.F. Demyanov, G.E. Stavroulakis, L.N. Polyakova, P.D. Panagiotopoulos, *Quasidifferentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics*, Kluwer Academic, Dordrecht, 1996. http://dx.doi.org/10.1007/978-1-4615-4113-4

[9] G. Di Pillo, L. Grippo, An exact penalty function method with global convergence properties for nonlinear programming problems, *Mathematical Programming*, **36** (1986), 1-18. http://dx.doi.org/10.1007/bf02591986

[10] G. Di Pillo, F. Facchinei, Regularity Conditions and Exact Penalty Functions in Lipschitz Programming Problems, *Nonsmooth Optimization Methods and Applications*, Ed. F. Gianessi, Gordon and Breach, Amsterdam, (1992), 107-120.

[11] S.P. Hen, O.L. Mangasarian, Exact penalty function in nonlinear programming, *Mathematical Programming*, **17** (1979), 251-269. http://dx.doi.org/10.1007/bf01588250

[12] J.B. Lasserre, A globally convergent algorithm for exact penalty functions, *European Journal of Operational Research*, **7** (1981), 389-395. http://dx.doi.org/10.1016/0377-2217(81)90097-7

[13] M. Mongeau, A. Sartenaer, Automatic decrease of the penalty parameter in exact penalty functions methods, *European Journal of Operational Research*, **83** (1995), 686-699. http://dx.doi.org/10.1016/0377-2217(93)e0339-y

[14] L.N. Polyakova, *Quasidifferentiable Optimization: Exact Penalty Methods, In: Encyclopedia of Optimization*, C. A. Floudas and P. M. Pardalos (eds.), Second edition, New York, 2009, 3200-3205. http://dx.doi.org/10.1007/978-0-387-74759-0_549

[15] L. Polyakova, V. Karelin, Exact penalty methods for nonsmooth optimization, *2014 20th International Workshop on Beam Dynamics and Optimization (BDO)*, (2014), 135-136. http://dx.doi.org/10.1109/bdo.2014.6890067

[16] E. Rosenberg, Globally convergent algorithms for convex programming, *Mathematics of Operational Research*, **6** (1981), 437-443. http://dx.doi.org/10.1287/moor.6.3.437

[17] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970. http://dx.doi.org/10.1515/9781400873173

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