Applied Mathematical Sciences, Vol. 8, 2014, no. 67, 3321 - 3330
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.44265

Homomorphism and Cartesian Product of Fuzzy PS – Algebras

T. Priya

Department of Mathematics
PSNA College of Engineering and Technology
Dindigul – 624 622, Tamilnadu, India

T. Ramachandran

Department of Mathematics
M.V.Muthiah Government Arts College for Women
Dindigul– 624 001, Tamilnadu, India

Abstract

In this paper, we introduce the concept of fuzzy PS-ideal of PS-algebra under homomorphism and some of its properties. We proved that β is a fuzzy PS-ideal(PS-subalgebra) of a PS-algebra X iff μ_β is a fuzzy PS-ideal(PS-subalgebra) of X x X, where μ_β is the strongest fuzzy β relation.

Mathematics Subject Classification: 20N25, 03E72, 03F055 , 06F35, 03G25

Keywords: PS-algebra, fuzzy PS-subalgebra, fuzzy PS- ideal, homomorphism, Cartesian product

1. Introduction

K.Iseki and S.Tanaka**[1,2]** introduced two classes of abstract algebras : BCK-algebras and BCI –algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J.Neggers, S.S.Ahn and H.S.Kim**[3]** introduced Q-algebras and d-algebras which is generalization of BCK / BCI algebras and obtained several results. C.Prabpayak and U.Leerawat **[4]** introduced a new algebraic structure which is called KU-algebras and investigated some properties. The concept of fuzzy set was introduced by L.A.Zadeh in 1965 **[14]**. Since then these ideas have been applied to other algebraic structures such as groups, rings, modules, vector spaces and topologies. T.Priya and T.Ramachandran [**5,6,7,8]** introduced the new algebraic structure, PS-algebra, which is an another generalization of BCI / BCK/Q /d/ KU algebras and investigated its properties related to fuzzy, fuzzy dot in detail. In this paper, We investigate the behavior of fuzzy PS-ideal and PS-subalgebra of PS-algebra, with the homomorphism and Cartesian products and obtain some of its results.

2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1 [5]

A nonempty set X with a constant 0 and a binary operation ‘ $*$ ‘ is called PS – Algebra if it satisfies the following axioms.

1. $x * x = 0$
2. $x * 0 = 0$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$, $\forall$ $x, y \in X$.

Example 2.1

Let X = { 0,1,2 } be the set with the following table

.

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

Then (X , $*$, 0) is a PS – Algebra.

Definition 2.2 [5,6]

Let X be a PS-algebra and I be a subset of X, then I is called a PS-ideal of X if it satisfies the following conditions:

1. $0 \in I$
2. $y * x \in I$ and $y \in I \Rightarrow x \in I$

Definition 2.3 [14]

Let X be a non-empty set. A fuzzy subset μ of the set X is a mapping $\mu : X \to [0,1]$.

Definition 2.4 [6,7]

Let X be a PS-algebra. A fuzzy set μ in X is called a fuzzy PS-ideal of X if it satisfies the following

conditions.

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu(y * x), \mu(y)\}$,for all $x,y \in X$

Definition 2.5 [6,8,9]

A fuzzy set μ in a PS-algebra X is called a fuzzy PS-subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$,

for all $x,y \in X$.

3. Homomorphism on Fuzzy PS-algebras

In this section, we discussed about Fuzzy PS-ideals and PS-subalgebra in PS-algebra under homomorphism and some of its properties.

Definition 3.1 [8,12,13]

Let $(X,*,0)$ and $(Y,*,0')$ be PS– algebras. A mapping $f: X \to Y$ is said to be a homomorphism if

$f(x * y) = f(x) * f(y)$ for all $x,y \in X$.

Remark:

If $f: X \to Y$ is a homomorphism of PS-algebra, then $f(0) = 0$.

Definition 3.2 [8,12]

Let $f: X \to X$ be an endomorphism and μ be a fuzzy set in X. We define a new fuzzy set in X by μ_f in X as $\mu_f(x) = \mu(f(x))$ for all x in X.

Theorem 3.3

Let f be an endomorphism of a PS- algebra X. If μ is a fuzzy PS-ideal of X, then so is μ_f.

Proof:

Let μ be a fuzzy PS-ideal of X.

Now, $\mu_f(x) = \mu(f(x)) \leq \mu(f(0)) = \mu_f(0)$, for all $x \in X$.

$\therefore \mu_f(0) \geq \mu_f(x)$

Let $x,y \in X$

Then $\mu_f(x) = \mu(f(x))$

$\geq \min\{\mu(f(y) * f(x)), \mu(f(y))\}$

$= \min\{\mu(f(y * x)), \mu(f(y))\}$

$= \min\{\mu_f(y * x), \mu_f(y)\}$

$\therefore \mu_f(x) \geq \min\{\mu_f(y * x), \mu_f(y)\}$

Hence μ_f is a fuzzy PS-ideal of X.

Theorem 3.4

Let f: $X \to Y$ be an epimorphism of PS- algebra. If μ_f is a fuzzy PS-ideal of X, then μ is a fuzzy PS-ideal of Y.

Proof:

Let μ_f be a fuzzy PS-ideal of X and let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$.

Now, $\mu(0) = \mu(f(0)) = \mu_f(0) \geq \mu_f(x) = \mu(f(x)) = \mu(y)$

$\therefore \mu(0) \geq \mu(y)$

Let $y_1, y_2 \in Y$. $\mu(y_1) = \mu(f(x_1)) = \mu_f(x_1)$

$\geq \min\{\mu_f(x_2 * x_1), \mu_f(x_2)\}$

$= \min\{\mu(f(x_2 * x_1)), \mu(f(x_2))\}$

$= \min\{\mu(f(x_2) * f(x_1)), \mu(f(x_2))\}$

$= \min\{\mu(y_2 * y_1), \mu(y_2)\}$

$\therefore \mu(y_1) \geq \min\{\mu(y_2 * y_1), \mu(y_2)\} \Rightarrow \mu$ is a fuzzy PS-ideal of Y.

Theorem 3.5

Let f: $X \to Y$ be a homomorphism of PS- algebra. If μ is a fuzzy PS-ideal of Y then μ_f is a fuzzy PS-ideal of X.

Proof:

Let μ be a fuzzy PS-ideal of Y and let $x,y \in X$.

Then $\mu_f(0) = \mu(f(0)) \geq \mu(f(x)) = \mu_f(x) \Rightarrow \mu_f(0) \geq \mu_f(x)$.

Also $\mu_f(x) = \mu(f(x))$

$\geq$ min { μ (f(y) * f(x)) , μ (f (y)) }
= min { μ (f(y * x)) , μ (f (y)) }
= min { μ_f (y * x) , μ_f (y) }

$\therefore$ μ_f (x) $\geq$ min { μ_f (y * x) , μ_f (y) }. Hence μ_f is a fuzzy PS-ideal of X.

Theorem 3.6

Let f : X→Y be a homomorphism of a PS-algebra X into a PS-algebra Y. If μ is a fuzzy PS- subalgebra of Y, then the pre- image of μ denoted by f^{-1} (μ), defined as {f^{-1} (μ)}(x) = μ(f(x)) , $\forall$ x $\in$ X, is a fuzzy PS- subalgebra of X.

Proof:

Let μ be a fuzzy PS- subalgebra of Y. Let x , y $\in$ X.

Now , {f^{-1} (μ)}(x* y) = μ (f (x *y))
= μ (f (x) * f(y))
$\geq$ min { μ (f (x)), μ(f(y)) }
= min {{f^{-1}(μ)} (x), {f^{-1}(μ)}(y) }

$\Rightarrow$ f^{-1} (μ) is a fuzzy PS-subalgebra of X.

Theorem 3.7

If μ be a fuzzy PS- subalgebra of X, then μ_f is also a fuzzy PS-subalgebra of X.

Proof:

Let μ_f be a fuzzy PS- subalgebra of X. Let x , y $\in$ X.

Now , μ_f (x* y) = μ (f (x *y))
= μ (f (x) * f(y))
$\geq$ min {μ (f (x)) , μ(f(y)) }
= min {μ_f (x) , μ_f (y)}

$\Rightarrow$ μ_f is a fuzzy PS-subalgebra of X.

4. Cartesian Product of Fuzzy PS-ideals of PS–algebras

In this section, we discuss the concept of Cartesian product of fuzzy PS-ideal and PS-subalgebra of PS-algebra.

Definition 4.1 [8]

Let μ and δ be the fuzzy sets in X. The Cartesian product μ x δ : X x X $\rightarrow$ [0,1] is defined by (μ x δ) (x, y) = min {μ(x),δ(y)}, for all x, y $\in$ X.

Theorem 4.2

If μ and δ are fuzzy PS-ideals in a PS– algebra X, then μ x δ is a fuzzy PS-ideal in X x X.

Proof:

Let $(x_1, x_2) \in X \times X$.

$(\mu \times \delta)(0,0) = \min\{\mu(0), \delta(0)\}$

$\geq \min\{\mu(x_1), \delta(x_2)\}$

$= (\mu \times \delta)(x_1, x_2)$

Let $(x_1, x_2), (y_1, y_2) \in X \times X$.

Now, $(\mu \times \delta)(x_1, x_2) = \min\{\mu(x_1), \delta(x_2)\}$

$\geq \min\{\min\{\mu(y_1 * x_1), \mu(y_1)\}, \min\{\delta(y_2 * x_2), \delta(y_2)\}\}$

$= \min\{\min\{\mu(y_1 * x_1), \delta(y_2 * x_2)\}, \min\{\mu(y_1), \delta(y_2)\}\}$

$= \min\{(\mu \times \delta)((y_1, y_2) * (x_1, x_2)), (\mu \times \delta)(y_1, y_2)\}$

$\therefore (\mu \times \delta)(x_1, x_2) \geq \min\{(\mu \times \delta)((y_1, y_2) * (x_1, x_2)), (\mu \times \delta)(y_1, y_2)\}$.

Hence, μ x δ is a fuzzy PS- ideal in X x X.

Theorem 4.3:

Let μ & δ be fuzzy sets in a PS-algebra X such that μ x δ is a fuzzy PS-ideal of X x X. Then

(i) Either $\mu(0) \geq \mu(x)$ (or) $\delta(0) \geq \delta(x)$ for all $x \in X$.

(ii) If $\mu(0) \geq \mu(x)$ for all $x \in X$, then either $\delta(0) \geq \mu(x)$ (or) $\delta(0) \geq \delta(x)$

(iii) If $\delta(0) \geq \delta(x)$ for all $x \in X$, then either $\mu(0) \geq \mu(x)$ (or) $\mu(0) \geq \delta(x)$

Proof:

Let μ x δ be a fuzzy PS-ideal of X x X .

(i) Suppose that $\mu(0) < \mu(x)$ and $\delta(0) < \delta(x)$ for some $x, y \in X$.

Then $(\mu \times \delta)(x, y) = \min\{\mu(x), \delta(y)\}$

$> \min\{\mu(0), \delta(0)\} = (\mu \times \delta)(0,0)$, Which is a contradiction.

Therefore $\mu(0) \geq \mu(x)$ or $\delta(0) \geq \delta(x)$ for all $x \in X$.

(ii) Assume that there exists $x, y \in X$ such that $\delta(0) < \mu(x)$ and $\delta(0) < \delta(x)$.

Then $(\mu \times \delta)(0,0) = \min\{\mu(0), \delta(0)\} = \delta(0)$ and hence

$(\mu \times \delta)(x, y) = \min\{\mu(x), \delta(y)\} > \delta(0) = (\mu \times \delta)(0,0)$ Which is a contradiction**.**

Hence, if $\mu(0) \geq \mu(x)$ for all $x \in X$, then either $\delta(0) \geq \mu(x)$ (or) $\delta(0) \geq \delta(x)$.

Similarly, we can prove that if $\delta(0) \geq \delta(x)$ for all $x \in X$, then either $\mu(0) \geq \mu(x)$ (or) $\mu(0) \geq \delta(x)$, which

yields (iii) .

Theorem 4.4

Let μ & δ be fuzzy sets in a PS-algebra X such that $\mu \times \delta$ is a fuzzy PS-ideal of X x X. Then either μ or δ is a fuzzy PS-ideal of X.

Proof:

First we prove that δ is a fuzzy PS-ideal of X.

Since by 4.3(i) either $\mu(0) \geq \mu(x)$ (or) $\delta(0) \geq \delta(x)$ for all $x \in X$.

Assume that $\delta(0) \geq \delta(x)$ for all $x\in X$. It follows from 4.3(iii) that either $\mu(0) \geq \mu(x)$ (or) $\mu(0) \geq \delta(x)$.

If $\mu(0) \geq \delta(x)$, for any $x\in X$, then $\delta(x) = \min \{\mu(0), \delta(x)\}= (\mu \times \delta) ((0, x)$

$$\begin{aligned}\delta(x) &= (\mu \times \delta) (0, x)\\ &\geq \min \{(\mu \times \delta)((0,y)* (0,x)), (\mu \times \delta) (0, y)\}\\ &= \min \{(\mu \times \delta) ((0*0),(y*x)), (\mu \times \delta) (0, y)\}\\ &= \min \{(\mu \times \delta) (0,(y*x)), (\mu \times \delta) (0, y)\}\\ &= \min \{ \delta(y*x), \delta(y)\}\end{aligned}$$

Hence δ is a fuzzy PS-ideal of X.

Next we will prove that μ is a fuzzy PS-ideal of X.

Let $\mu(0) \geq \mu(x)$.Since by theorem 4.3(ii), either $\delta(0) \geq \mu(x)$ (or) $\delta(0) \geq \delta(x)$.

Assume that $\delta(0) \geq \mu(x)$, then $\mu(x) = \min \{ \mu(x) , \delta(0)\} = (\mu \times \delta) (x,0)$

$$\begin{aligned}\mu(x) &= (\mu \times \delta) (x ,0)\\ &\geq \min\{(\mu \times \delta)((y,0) * (x,0)), (\mu \times \delta) (y,0)\}\\ &= \min \{(\mu \times \delta) ((y * x), (0*0)), (\mu \times \delta) (y,0)\}\\ &= \min \{(\mu \times \delta) ((y * x), 0), (\mu \times \delta) (y,0)\}\\ &= \min \{\mu (y * x), \mu(y)\}\end{aligned}$$

Hence μ is a fuzzy PS-ideal of X.

Theorem 4.5

If λ and μ are fuzzy PS-subalgebras of a PS-algebra X , then $\lambda \times \mu$ is also a fuzzy PS-subalgebra of X x X.

Proof :

For any $x_1, x_2, y_1, y_2 \in X$.

$$(\lambda \times \mu)((x_1, y_1) * (x_2, y_2)) = (\lambda \times \mu)(x_1 * x_2, y_1 * y_2)$$
$$= \min \{ \lambda(x_1 * x_2), \mu(y_1 * y_2) \}$$
$$\geq \min \{ \min \{ \lambda(x_1), \lambda(x_2) \}, \min\{ \mu(y_1), \mu(y_2) \} \}$$
$$= \min \{ \min \{ \lambda(x_1), \mu(y_1) \}, \min \{ \lambda(x_2), \mu(y_2) \} \}$$
$$= \min \{ (\lambda \times \mu)(x_1, y_1), (\lambda \times \mu)(x_2, y_2) \}$$

This completes the proof.

Definition 4.6

Let β be a fuzzy subset of X. The strongest fuzzy β-relation on PS-algebra X is the fuzzy subset μ_β of X x X given by $\mu_\beta(x, y) = \min\{ \beta(x), \beta(y) \}$, for all $x, y \in X$.

Theorem 4.7

Let μ_β be the strongest fuzzy β-relation on PS-algebra X , where β is a fuzzy set of a PS-algebra X. If β is a fuzzy PS-ideal of a PS-algebra X , then μ_β is a fuzzy PS-ideal of X x X.

Proof :

Let β be a fuzzy PS-ideal of a PS-algebra X.

Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then $\mu_\beta(0,0) = \min \{ \beta(0), \beta(0) \} \geq \min \{ \beta(x_1), \beta(x_2) \} = \mu_\beta(x_1, x_2)$

and also $\mu_\beta(x_1, x_2) = \min \{ \beta(x_1), \beta(x_2) \}$

$$\geq \min \{ \min \{ \beta(y_1 * x_1), \beta(y_1) \}, \min \{ \beta(y_2 * x_2), \beta(y_2) \} \}$$
$$= \min \{\min \{ \beta(y_1 * x_1), \beta(y_2 * x_2) \}, \min \{\beta(y_1), \beta(y_2)\}\}$$
$$= \min \{ \mu_\beta((y_1 * x_1), (y_2 * x_2)), \mu_\beta(y_1, y_2) \}$$
$$= \min \{ \mu_\beta((y_1, y_2) * (x_1, x_2)), \mu_\beta(y_1, y_2) \}$$

Therefore μ_β is a fuzzy PS-ideal of X x X.

Theorem 4.8

If μ_β is a fuzzy PS-ideal of X x X, then β is a fuzzy PS-ideal of a PS-algebra X.

Proof :

Let μ_β is a fuzzy PS-ideal of X x X. Then for all $(x_1, x_2), (y_1, y_2) \in X \times X$.

$\min \{ \beta(0), \beta(0) \} = \mu_\beta(0,0) \geq \mu_\beta(x_1, x_2) = \min \{ \beta(x_1), \beta(x_2) \}$

$\Rightarrow \beta(0) \geq \beta(x_1)$ or $\beta(0) \geq \beta(x_2)$.

Also, $\min \{ \beta(x_1), \beta(x_2) \} = \mu_\beta(x_1, x_2)$

$$\geq \min \{ \mu_\beta((y_1, y_2) * (x_1, x_2)), \mu_\beta(y_1, y_2) \}$$
$$= \min \{ \mu_\beta((y_1 * x_1), (y_2 * x_2)), \mu_\beta(y_1, y_2) \}$$
$$= \min \{\min \{ \beta(y_1 * x_1), \beta(y_2 * x_2) \}, \min \{\beta(y_1), \beta(y_2)\}\}$$
$$= \min \{ \min \{ \beta(y_1 * x_1), \beta(y_1) \}, \min \{ \beta(y_2 * x_2), \beta(y_2) \} \}$$

Put $x_2 = y_2 = 0$, we get $\beta(x_1) \geq \min \{ \beta(y_1 * x_1), \beta(y_1) \}$

Hence β is a fuzzy PS-ideal of a PS-algebra X.

Theorem 4.9

If β is a fuzzy PS-subalgebra of a PS-algebra X , then μ_β is a fuzzy PS-subalgebra of X x X.

Proof :

Let β be a fuzzy PS-subalgebra of a PS-algebra X and let $x_1, x_2, y_1, y_2 \in X$.

$$\begin{aligned} \text{Then } \mu_\beta((x_1, y_1) * (x_2, y_2)) &= \mu_\beta(x_1 * x_2, y_1 * y_2) \\ &= \min\{\beta(x_1 * x_2), \beta(y_1 * y_2)\} \\ &\geq \min\{\min\{\beta(x_1), \beta(x_2)\}, \min\{\beta(y_1), \beta(y_2)\}\} \\ &= \min\{\min\{\beta(x_1), \beta(y_1)\}, \min\{\beta(x_2), \beta(y_2)\}\} \\ &= \min\{\mu_\beta(x_1, y_1), \mu_\beta(x_2, y_2)\} \end{aligned}$$

Therefore μ_β is a fuzzy PS-subalgebra of X x X.

Theorem 4.10

If μ_β is a fuzzy PS-subalgebra of X x X, then β is a fuzzy PS-subalgebra of a PS-algebra X.

Proof :

Let $x, y \in X$.

$$\begin{aligned} \text{Now, } \beta(x * y) = \min\{\beta(x * y), \beta(x * y)\} &= \mu_\beta((x * y) * (x * y)) \\ &\geq \min\{\mu_\beta(x * y), \mu_\beta(x * y)\} \\ &= \min\{\min\{\beta(x), \beta(y)\}, \min\{\beta(x), \beta(y)\}\} \\ &= \min\{\beta(x), \beta(y)\} \end{aligned}$$

$\Rightarrow \beta(x * y) \geq \min\{\beta(x), \beta(y)\}$, which completes the proof.

5. Conclusion

In this article authors have been discussed fuzzy PS-ideals and PS-subalgebras in fuzzy PS-algebra under homomorphism and Cartesian product. It has been observed that PS-algebra as a generalization of BCK/BCI/Q/d/TM/KU-algebras. Interestingly, the strongest fuzzy β-relation on PS-algebra concept has been discussed in Cartesian products and it adds an another dimension to the defined fuzzy PS--algebra. This concept can further be generalized to Intuitionistic fuzzy set, interval valued fuzzy sets, Anti fuzzy sets for new results in our future work.

Acknowledgement

Authors wish to thank Dr. K.T.Nagalakshmi ,Professor and Head, Department of Mathematics, KLN College of Information and Technology, Pottapalayam, Sivagangai District, Tamilnadu, India.

Prof. P.M. Sithar Selvam, Department of Mathematics, PSNA College of Engineering and Technology, Dindigul, Tamilnadu , India and family members for their help to make this paper as successful one.

References

[1] K.Iseki and S.Tanaka , An introduction to the theory of BCK – algebras ,Math Japonica 23 (1978), 1- 20

[2] K.Iseki , On BCI-algebras , Math.Seminar Notes 8 (1980), 125-130.

[3] J.Neggers and H.S.Kim , On d-algebras , Math,slovaca, 49 (1999) , 19-26.

[4] C.Prabpayak andU.Leerawat,On Ideals and congurences in KU-algebras,Scientia Magna J. 5(1)(2009),54-57.

[5] T.Priya and T.Ramachandran ,Classification of PS-algebras, The International Journal of Science and Technology, Communicated.

[6] T.Priya and T.Ramachandran ,A note on Fuzzy PS –Ideals in PS-Algebras, International Journal of Advanced Mathematical Sciences, in press.

[7] T.Priya and T.Ramachandran , Some properties of fuzzy dot PS-sub algebras of PS-algebras, Annals of Pure and Applied Mathematics,Vol.6, No.1, 2014, 11-18**.**

[8] T.Priya and T.Ramachandran , Some Characterization of Anti fuzzy PS-ideals of PS-algebras in Homomorphism and Cartesian Products , International Journal of Fuzzy Mathematical Archive, Vol.4, No.2 , 2014 ,72-79.

[9] T.Priya and T.Ramachandran, Anti fuzzy PS-ideals in PS-algebras and its level subsets , International Journal of Mathematical Archive ,Vol.5, No.4 , 2014 , 1-7.

[10] T.Priya and T.Ramachandran, Normalisation of fuzzy PS-algebras, International Journal of Engineering Research and Applications, Communicated.

[11] T.Priya and T.Ramachandran, On Pseudo fuzzy PS-algebras, Annals of Pure and Applied Mathematics Communicated.

[12] P.M.Sithar Selvam, T.Priya and T.Ramachandran , Anti Fuzzy Subalgebras and Homomorphism of CI-algebras, International Journal of Engineering Research & Technology,Vol. 1 Issue 5, July – 2012,1-6.

[13] P.M.Sithar Selvam and K.T.Nagalakshmi, A Note on Anti Fuzzy Sub-Bi HX Group and its Bi Level Sub-Bi HX Groups, Applied Mathematical Sciences, Volume 8(39),2014,1925-1932.

[14] Zadeh. L.A.Fuzzy sets, Information and Control,8, 1965, 338-353.

Received: April 5, 2014