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# Global Convergence Properties of a New Class of Conjugate Gradient Method for Unconstrained Optimization

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**Abstract**

Nonlinear conjugate gradient (CG) methods are widely used for solving large scale unconstrained optimization problems. Many studies have been devoted to modified and improve this method. In this paper, a new parameter of CG method that possesses global convergence properties using exact line search is proposed. Numerical results show that the new formula is best and more efficient when compared with the other classical CG methods.

**Mathematics Subject Classification:** 65K10, 49M37

**Keywords**: Conjugate gradient, exact line search, global convergence

## 1. Introduction

Consider the following unconstrained optimization problem

$$\min_{x \in R^n} f(x) \tag{1}$$

where $f: R^n \to R$ is continuously differentiable function, the CG methods are the best methods for solving (1), especially when the dimension *n* is large. The iterates of CG methods for solving (1) are obtained by

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 0,1,2,\dots, \tag{2}$$

where $x_k$ is current iterate point and $\alpha_k > 0$ is step size. The step size is computed by carrying out some line search, especially the exact line search

$$f(x_k + \alpha_k d_k) = \min_{\alpha \geq 0} f(x_k + \alpha d_k) \tag{3}$$

The search direction $d_k$ is defined by

$$d_k = \begin{cases} -g_k & \text{if} \quad k = 0, \\ -g_k + \beta_k d_{k-1} & \text{if} \quad k \geq 1, \end{cases} \tag{4}$$

where $\beta_k \in R$ is a scalar. We know that some classical formula's for $\beta_k$ are the Hestenes-Stiefel (HS) in1952, the Fletcher-Reeves (FR) in1964, the Polak-Ribiere- Polyak (PRP) in 1969, the conjugate descent method (CD) in 1987, the Liu-Storey (LS) in 1992, and the Dai-Yuan (DY) in 2000. The parameters of these $\beta_k$ are given as follows

$$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{(g_k - g_{k-1})^T d_{k-1}}, \qquad \beta_k^{FR} = \frac{g_k^T g_k}{\|g_{k-1}\|^2}, \qquad \beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2},$$

$$\beta_k^{CD} = -\frac{g_k^T g_k}{d_{k-1}^T g_{k-1}}, \qquad \beta_k^{LS} = \frac{g_k^T(g_k - g_{k-1})}{-d_{k-1} g_{k-1}}, \qquad \beta_k^{DY} = \frac{g_k^T g_k}{(g_k - g_{k-1})^T d_{k-1}}.$$

The convergence properties of the FR conjugate gradient method have been studied by many authors, such as Al-Baali (1985)and Liu et al. (1990). Some researchers have also studied the convergence properties of the other classical formulas. For instance, Powell (1984) and Zoutendijk(1970) proved that the FR method with exact line searches is globally convergent on general functions. Al-Baali(1985) extended this result to inexact line searches, Touati-Ahmed and Storey(1990), Gilbert and Nocedal (1992) gave another way to discuss the global convergence of the PRP method with the weak Wolfe-Powell line search. In this case, the parameter $\beta_k$ is not allowed to be negative that is

$$\beta_k = \max\left\{\beta_k^{PRP}, 0\right\}$$

Some researchers have also presented good comparative study of some new CG methods such as Andrei (2009), Rivaie et al.(2012), Sun and Zhang (2001), Wei et al. (2006), Farid et al.(2013), Jusoh et al.(2013), and lastly Hager and Zhang (2005).

In this paper, we will show our new $\beta_k$ and its algorithm in section 2. While in section 3, we show the sufficient descent condition and the global convergence proof of our new method. Later on in section 4,we deals with the Numerical results and discussion and finally we wrapped up everything as a conclusion in section 5.

## 2. New $\beta_k$ and Algorithm

In mid of 2012, Rivaie et al. proposed a new nonlinear CG formula which is simple and easy to be used. This formula is given as

$$\beta_k^{RMIL} = \frac{g_k^T\left(g_k - g_{k-1}\right)}{\left\|d_{k-1}\right\|^2}. \tag{5}$$

Even though this formula is simple, they have manage to show that this formula possessed four main requirements of CG formula, the sufficient descent conditions, global convergence properties, angle conditions and linear convergence rate. Intrigued by this new findings, we proposed a new modification of $\beta_k$ which is defined by

$$\beta_k^{AMRI} = \frac{\left\|g_k\right\|^2 - \dfrac{\left\|g_k\right\|}{\left\|g_{k-1}\right\|}\left|g_k^T g_{k-1}\right|}{\left\|d_{k-1}\right\|^2}. \tag{6}$$

The AMRI denotes Abashar, Mustafa, Rivaie and Ismail. It is clear that this new formula possessed the same denominator as RMIL.The following algorithm is a general algorithm for CG methods.

**Algorithm 2.1**

Step 1: Initialization. Given $x_0$, set $k = 0$.

Step 2: Compute $\beta_k$, based on predetermined formula

Step 3: Compute $d_k$ based on (4). If $g_k = 0$, then stop.

Step 4: Compute $\alpha_k$ based on exact line search (3).

Step 5: Updating new point based on iterative formula (2).

Step 6: Convergent test and stopping criteria.

If $f(x_{k+1}) < f(x_k)$ and $\|g_k\| \le \varepsilon$ then stop.

Otherwise go to Step 1 with $k = k+1$.

## 3. Global Convergence analysis

In this section, we will show the global convergent properties of $\beta_k^{AMRI}$. We first begin with the sufficient descent condition.

### 3.1. Sufficient descent condition

The sufficient descent condition, which is express as,

$$g_k^T d_k \le -C\|g_k\|^2 \text{ for } k \ge 0,\ C > 0 \tag{7}$$

The following theorem shows that AMRI with exact line search possess the sufficient descent condition.

**Theorem 1.**

Suppose that the $x_k$ and $d_k$ are generated by the method of the form (2), (4) and (6), and the step size $\alpha_k > 0$ determined by the exact line search then, condition (7) holds for all $k \ge 0$

**Proof:**

The proof is by Induction. If $k = 0$ then we already have $g_0^T d_0 = -C\|g_0\|^2$. Hence condition (7) holds true. Then to continue, we need to show that for $k \ge 1$, condition (7) will also holds true. From (4) multiply by $g_{k+1}^T$ then

$$g_{k+1}^T d_{k+1} = g_{k+1}^T(-g_{k+1} + \beta_{k+1} d_k) = -\|g_{k+1}\|^2 + \beta_{k+1} g_{k+1}^T d_k \,. \tag{8}$$

For exact line search, we know that $g_{k+1}^T d_k = 0$. Thus

$$g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2 \,.$$

Hence, this condition holds true for $k+1$. The proof is completed.⦿

### 3.2. Global convergence properties

To study the global convergence properties, first we must prove that $\beta_k^{AMRI}$ are always not less than zero

$$\beta_{k+1}^{AMRI} = \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|}\left|g_{k+1}^T g_k\right|}{\|d_k\|^2} \geq \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|}\|g_{k+1}\|\|g_k\|}{\|d_k\|^2} = 0 \tag{9}$$

We can simplify $\beta_{k+1}^{AMRI}$

$$\beta_{k+1}^{AMRI} = \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|}\left|g_{k+1}^T g_k\right|}{\|d_k\|^2} \leq \frac{\|g_{k+1}\|^2}{\|d_k\|^2} \tag{10}$$

In many theorem proofs, we also needed the following assumption

**Assumption 1**

(i) The level set $\Omega = \left\{ x \in R^n \,\middle|\, f(x) \leq f(x_0) \right\}$ is bounded, where $x_0$ is the starting point.

(ii) In some neighborhood $N$ of, $\Omega$ the objective function is continuously differentiable, and its gradient is Lipschitz continuous, namely, there exists a constant $l > 0$ such that $\|g(x) - g(y)\| \leq l\|x - y\|$ for any $x, y \in N$.

Under this assumption, we have the following lemma, which was proved by Zoutendijk(1970).

**Lemma 1.**

Suppose Assumption 1 holds, let $x_k$ be generated by Algorithm 2.1 and $d_k$ satisfy (7) then the following condition, known as the Zoutendijk condition, holds

$$\sum_{k=0}^{\infty} \frac{\left(g_k^T d_k\right)^2}{\|d_k\|} < +\infty \tag{11}$$

The following theorem is based on Lemma 1.

**Theorem 2.**
Suppose that Assumption 1 holds true, $x_k$ is generated by Algorithm 2.1, the $\alpha_k$ is obtained by the exact line search (3) and the sufficient descent condition hold true. Then either

$$\lim_{k\to\infty}\|g_k\| = 0 \qquad \text{or} \qquad \sum_{k=0}^{\infty}\frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty .$$

**Proof:**
To prove Theorem 2, we use contradiction. That is, if Theorem 2 is not true, then a constant $c > 0$ exists, such that

$$\|g_k\| \geq c \tag{12}$$

Rewriting (4) as

$$d_{k+1} + g_{k+1} = \beta_{k+1} d_k$$

And squaring both sides of the equation, we get

$$\|d_{k+1}\|^2 = (\beta_{k+1})^2\|d_k\|^2 - 2g_{k+1}^T d_{k+1} - \|g_{k+1}\|^2 \tag{13}$$

Dividing both side by $\left(g_{k+1}^T d_{k+1}\right)^2$ then,

$$\frac{\|d_{k+1}\|^2}{\left(g_{k+1}^T d_{k+1}\right)^2} = \frac{(\beta_{k+1})^2\|d_k\|^2}{\left(g_{k+1}^T d_{k+1}\right)^2} - \frac{2}{g_{k+1}^T d_{k+1}} - \frac{\|g_{k+1}\|^2}{\left(g_{k+1}^T d_{k+1}\right)^2}$$

$$= \frac{(\beta_{k+1})^2\|d_k\|^2}{\left(g_{k+1}^T d_{k+1}\right)^2} - \left(\frac{1}{\|g_{k+1}\|} + \frac{\|g_{k+1}\|}{g_{k+1}^T d_{k+1}}\right)^2 + \frac{1}{\|g_{k+1}\|^2}$$

$$\leq \frac{(\beta_{k+1})^2\|d_k\|^2}{\left(g_{k+1}^T d_{k+1}\right)^2} + \frac{1}{\|g_{k+1}\|^2}$$

Applying (9) we get

$$\leq \frac{1}{\|g_{k+1}\|^2} \tag{14}$$

Hence

$$\frac{\|d_k\|^2}{\left(g_k^T d_k\right)^2} \leq \sum_{i=0}^{k}\frac{1}{\|g_i\|^2}$$

$$\frac{\left(g_k^T d_k\right)^2}{\|d_k\|^2} \geq \frac{c^2}{k} \tag{15}$$

There from (15) and (12), it follows that

$$\sum_{k=0}^{\infty}\frac{\left(g_k^T d_k\right)^2}{\|d_k\|^2}=\infty$$

This contradicts the Zoutendijk condition in Lemma 1.The proof is completed.⦿

**Theorem 3**

Suppose that Assumptions 1 holds, consider any CG the methods of form (2) and (4), the $\alpha_k$ obtained by the exact line search and $\beta_k$ is determined by (6). Then either

$$\lim_{k\to\infty}\|g_k\|=0 \text{ or } \sum_{k=0}^{\infty}\frac{(g_k^T)^4}{\|d_k\|^2}<\infty .$$

**Proof:**

From (13) and (10)

$$\|d_{k+1}\|^2=\left(\frac{\|g_{k+1}\|^2}{\|d_k\|^2}\right)^2\|d_k\|^2-2g_{k+1}^T d_{k+1}-\|g_{k+1}\|^2$$

$$\|d_{k+1}\|^2=\frac{\|g_{k+1}\|^4}{\|d_k\|^2}-2g_{k+1}^T d_{k+1}-\|g_{k+1}\|^2 \tag{16}$$

We have already proven that sufficient descent condition holds. Therefore, we know that

$$g_{k+1}^T d_{k+1}\le -C\|g_{k+1}\|^2 .$$

Hence from (16),

$$\|d_{k+1}\|^2=\frac{\|g_{k+1}\|^4}{\|d_k\|^2}+2c\|g_{k+1}\|^2-\|g_{k+1}\|^2$$

$$\|d_{k+1}\|^2=\frac{\|g_{k+1}\|^4}{\|d_k\|^2}-\|g_{k+1}\|^2(1-2c) \tag{17}$$

Multiply both sides of (17) with $\frac{\|g_{k+1}\|^2}{\|d_{k+1}\|^2}$ then,

$$\|d_{k+1}\|^2\frac{\|g_{k+1}\|^2}{\|d_{k+1}\|^2}=\frac{\|g_{k+1}\|^2}{\|d_{k+1}\|^2}\left(\frac{\|g_{k+1}\|^4}{\|d_k\|^2}-\|g_{k+1}\|^2(1-2c)\right)$$

$$\frac{\|d_{k+1}\|^2\|g_{k+1}\|^2}{\|d_{k+1}\|^2}=\frac{\|g_{k+1}\|^4}{\|d_{k+1}\|^2}\left((2c-1)+\frac{\|g_{k+1}\|^2}{\|d_k\|^2}\right)$$

$$\frac{\|d_{k+1}\|^2 \|g_{k+1}\|^2}{\|d_{k+1}\|^2} \leq \frac{\|g_{k+1}\|^4}{\|d_{k+1}\|^2} \quad (18)$$

Based on Theorem 2, we know that $\lim_{k\to\infty} \frac{(g_{k+1}^T d_{k+1})^2}{\|d_{k+1}\|} < 0$.

This will imply that if theorem 3 is not true, then we have $\lim_{k\to\infty} \frac{(g_{k+1}^T d_{k+1})^2}{\|d_{k+1}\|} = \infty$ and from (18) we get $\infty \leq \frac{\|g_{k+1}\|^2}{\|d_{k+1}\|^2}$. Hence, Theorem 3 holds true for sufficiently large $k$.

## 4. Numerical results and discussions

In this section, we present some numerical results. Twentyone test problems considered in Andrei (2008)have been selected as shown in table 1, which ranges from small scale to large scale to analyze efficiency of AMRI. Based on this, a comparison was evaluated with the other CG methods which include FR, PRPand RMIL. We considered $\varepsilon = 10^{-6}$ and the gradient values as the stopping criteria base on the work of Hillstrom(1977) who suggested that the stopping criteria $\|g_k\| \leq \varepsilon$. For each of the test functions problem, four different initial points are utilized, taking from the one that is closer to the solution and moving to the one that is furthest from it. Therefore, these four initial points will lead us to test to global convergence properties of our method. All problems are tested using MATLAB version 7.10.0 (R 2010a) subroutine programming. We used the exact line search to overcome the complexity of the algorithm as well as how to achieve the exact value of step size value. The CPU processor used was Core™ i3-2328M(2.2GHZ,3MB L3 Cache), with 6GB DDR3 RAM. In some cases failure occurs when the line search were unable to find the positive step size. Numerical results are compared relatively on the number of iteration and CPU time. The performance results shown in Figure1 and 2, respectively, using a performance profile introduced by Dolan and More (2002).

The AMRI method shows the best performance, as it can solve all of the test problems perfectly as shown in Figure 1 and 2. These conjugate gradient coefficients could also be divided into three types. The first type consists of PRP and the second type consists of FR and third type RMIL method and our new method AMRI. We can say that the first type possesses the restart properties, and their performance is much better, the second type does not possess this property, and therefore is slower.

**Table 1.** A list of problem function

| No | Functions | Variables | Initial point |
|---|---|---|---|
| 1 | Six hump Camel | 2 | (8,8),(-8,-8),(10,10),(-10,-10) |
| 2 | Booth | 2 | (10,10),(25,25),(50,50),(100,100) |
| 3 | Treccani | 2 | (5,5),(10,10),(50,50),(100,100) |
| 4 | Zettl | 2 | (5,5),(10,10),(20,20),(50,50) |
| 5 | Rosenbrock | 2,4,10,100,500 ,1000,10000 | (13,13,…,13),(16,16,…,16),(20,20,…,20),(30,30,…, 30) |
| 6 | Extended Penalty | 2,4,10,100 | (3,3,…,3),(6,6,…,6),(10,10,…,10),(30,30,…,30) |
| 7 | Extended Beale | 2,4,10,100,500 ,1000,10000 | (1,1,…1),(3,3,…,3),(7,7,…,7),(10,10,…,10) |
| 8 | Shallow | 2,4,10,100,500 ,1000,10000 | (10,10,…,10),(25,25,…,25),(50,50,…,50),(100,100, …,100) |
| 9 | Ex-Tridiagonal 1 | 2,4,10,100,500 ,1000,10000 | (30,30,…,30),(12,12,…,12),(17,17,…,17),(20,20,…, 20) |
| 10 | Raydan 1 | 2,4,10,100 | (1,1,…,1),(3,3,…,3),(5,5,…,5),(-10,-10,…,-10) |
| 11 | White and Holst | 2,4,10,100,500 ,1000,10000 | (3,3,…,3),(6,6,…,6),(9,9,…,9),(-3,-3,…,-3) |
| 12 | Quadrtic QF2 | 2,4,10,100,500 ,1000 | (5,5,…,5),(7,7,…,7),(10,10,…,10),(-3,-3,…,-3) |
| 13 | Diagonal 4 | 2,4,10,100,500 ,1000,10000 | (2,2,…,2),(5,5,…,5),(10,10,…,10),(15,15,…,15) |
| 14 | Extended Denschnb | 2,4,10,100,500 ,1000,10000 | (5,5,…,5),(8,8,…,8),(13,13,…,13),(25,25,…,25) |
| 15 | Hager | 2,4,10,100 | (3,3,…,3),(10,10,…,10),(15,15,…,15),(30,30,…,30) |
| 16 | Generalized Tridiagonal 1 | 2,4,100 | (2,2,…,2),(5,5,…,5),(15,15,…,15),(25,25,…,25) |
| 17 | Nonscomp | 2,4,100,500 | (3,3,…,3),(10,10,…,10),(13,13,…,13),(15,15,…,15) |
| 18 | Perturbed Quadratic | 2,4,10,100 | (3,3,…,3),(10,10,…,10),(15,15,…,15),(30,30,…,30) |
| 19 | Diagonal 2 | 2,4,10,100,500 ,1000 | (5,5,…,5),(10,10,…,10),(15,15,…,15),(20,20,…,20) |
| 20 | Quadratic Penalty QP2 | 2,4,10,100,500 | (10,10,…,10),(20,20,…,20),(50,50,…,50), (100,100,…,100) |
| 21 | Himmelbau | 2,4,10,100,500 ,1000,10000 | (10,10,…,10),(50,50,…,50),(100,100,…,100),(200,2 00,…,200) |

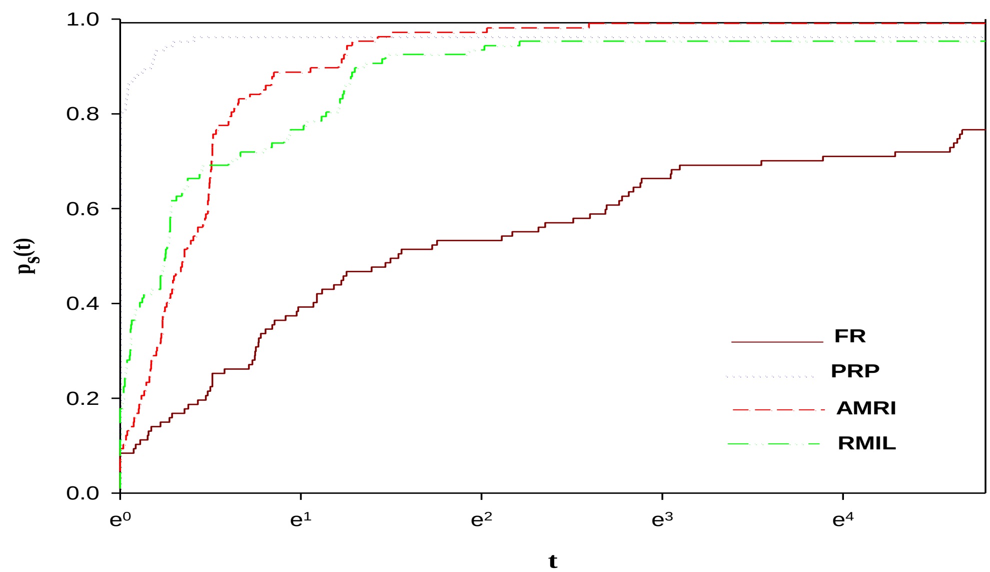


**Figure 1** Performance profile based on the number of iterations.

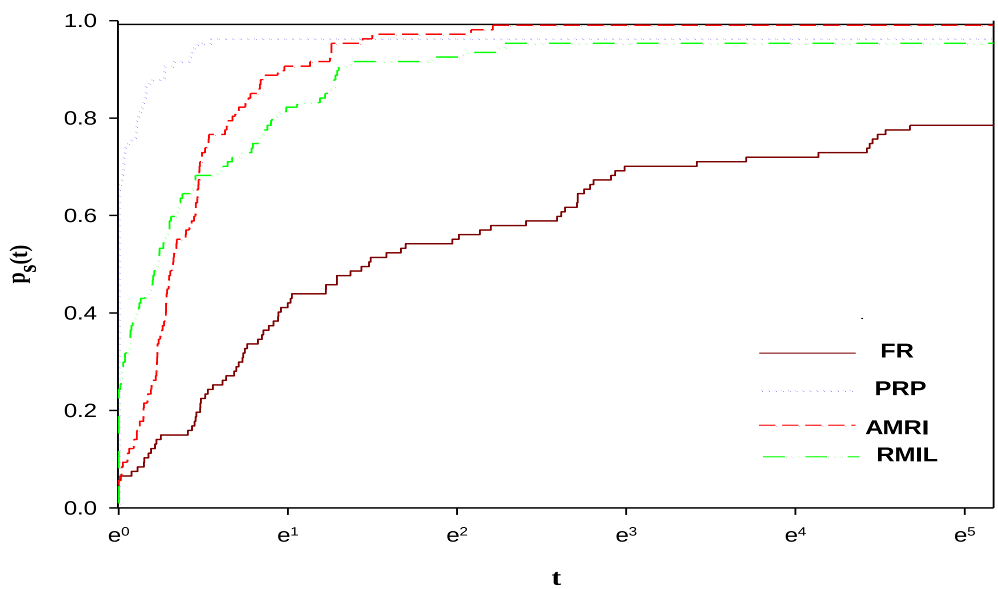


**Figure 2** Performance profile based on the CPU time.

Although, the performance of the first type is faster than AMRI, butit can solve only 96% of the test function problems while the second type solves only78% of the test function problems, and

RMIL method solve 95% of the test function problems. The performance of AMRI therefore lies between FR and PRP much nearer to the PRP method. We show that AMRI is best for other methods because it can solve all of the test function problems.

## 5. Conclusion

So many researchers have been done previously on CG method which led to the discovery of a variety of CG methods. In the same manner, our paper proposed a new and simple $\beta_k$ that is easy to be implemented and possesses the global convergence properties. The numerical results have shown that our new method has the best performance when compared to other standard CG methods. For our next research, we are hoping that this new formula could still be improved by the used of inexact line search or by the introduction of new scaling factors.

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