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Mathematical Modeling of Nuclear Family and Stability Analysis

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Abstract

In this paper we first give a new integer order mathematical model of nuclear family. Then, we offer a fractional order mathematical model and investigate the stability of solutions. Finally, we give examples to illustrate our results.

Keywords: Nuclear family model, Fractional differential equations, Initial value problem, Stability, Numerical solutions

1 Introduction

Family can be defined in many different ways. In its most basic terms, a family is a group of individuals who share a legal or genetic bond. There are many different types of families. The nuclear family is the traditional type of family structure. This family type consists of two parents and children. Children in nuclear families receive strength and stability from the two-parent structure and generally have more apportunities due to the financial ease of two adults. According to U.S. Cencus data, almost 70 percent of children live in a nuclear family unit.

Mathematical models capturing the dynamics of people have recently gained attention among many researchers. Strogatz presented a model on love affairs [13]. As an extension to Strogatz's seminal model, Rinaldi proposed a model with taking into account the appeal that each partner presents to the others in absence of other feelings [11]. Also Rapoport, Radzicki, Gottmann, Sprott, Wauer have recently proposed dynamical models on love affairs [4,9,10,12,14].

Besides the models of integer order, fractional order models are also discussed. Generally models offer a generalization of a dynamical models recently reported in the literature. The generalization is obtained by permitting the state dynamics of the model to assume fractional orders [7,5,2].

In the line of the aforementioned models, we have shown other applications for the dynamics of people model. We also showed that the model can be applied to a triadic interaction. In this paper we first give a new integer order mathematical model of nuclear family. Then, we offer a fractional order mathematical model and investigate the stability of solutions.

2 Model Derivation

2.1 Integer Order Model

In this section, integer order nuclear family model is introduced with four state variables. The model describes baby's emotions, in which baby (B) is involved in emotions with mother (M) and father (F). We use the following notation for variables:

$B(t)$: Baby's love for the baby's father,
$F(t)$: Father's love for the baby and his wife,
$M(t)$: Mother's love for the baby and her husband,
$B_1(t)$: Baby's love for the baby's mother.

The integer order nuclear family model is given as

$$\begin{aligned} \frac{dB}{dt} &= aB + b(F-M)(c-(F-M)) + \gamma_1 \\ \frac{dF}{dt} &= eF + gB(h-B) + jM + \gamma_2 \\ \frac{dM}{dt} &= kM + mB_1(n-B_1) + pF + \gamma_3 \\ \frac{dB_1}{dt} &= aB_1 + b(M-F)(d-(M-F)) + \gamma_4 \end{aligned} \tag{1}$$

with initial conditions

$$B(0) = B_0,\ F(0) = F_0,\ M(0) = M_0,\ B_1(0) = B_{10} \tag{2}$$

where

a, b, c, d: specify baby's emotional style,
e, g, h, j: specify father's emotional style,
k, m, n, p: specify mother's emotional style,
$\gamma_1, \gamma_2, \gamma_3, \gamma_4$: attraction constants.

2.2 Fractional Order Model

Modeling is becoming an increasingly important tool in many branches of mathematic. Mathematical modeling is the use of mathematics to describe real-world phenomena, investigate important questions about the observed world, make predictions and etc. In recent years, fractional order modeling is more popular than integer order modeling. Models have been used to describe dynamic phenomena in a wide range of fields, ranging from physical, natural, biological to social sciences. If fractional order models have been considered instead of its integer order counterpart, it can be more appropriate for this such dynamics. Therefore, many models have been considered as fractional order.

We begin by giving the definitions of fractional order integrals and derivatives [8].

Definition 1 *The Riemann-Liouville type fractional integral of order* $\alpha > 0$ *of a function* $f : (0, \infty) \to R$ *is defined by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Here and elsewhere Γ *denotes the Gamma function.*

Definition 2 *The Riemann-Liouville type fractional derivative of order* $\alpha > 0$ *of a function* $f : (0, \infty) \to R$ *is defined by*

$$\mathbf{D}^{\alpha} f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau$$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *is the integer part of* α.

For fractional order differentiation, we will use Caputo's definition, due to its convenience for initial conditions of the differential equations. Also Caputo differential operator is convenient for modeling since the derivative of a constant is zero which means that this kind of derivative can be used to model the rate of chance.

Definition 3 *The Caputo type fractional derivative of order* $\alpha > 0$ *of a function* $f : (0, \infty) \to R$ *is defined by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau$$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *is the integer part of* α.

In system (1), when we change the order of the equation to α, the dimension of the left-hand side would be $(\text{time})^{-\alpha}$. Then fractional order model given by (3) where $0 < \alpha \leq 1$.

$$\begin{aligned} \frac{d^\alpha B}{dt} &= aB + b(F-M)(c-(F-M)) + \gamma_1 \\ \frac{d^\alpha F}{dt} &= eF + gB(h-B) + jM + \gamma_2 \\ \frac{d^\alpha M}{dt} &= kM + mB_1(n-B_1) + pF + \gamma_3 \\ \frac{d^\alpha B_1}{dt} &= aB_1 + b(M-F)(d-(M-F)) + \gamma_4 \end{aligned} \tag{3}$$

with initial conditions

$$B(0) = B_0,\ F(0) = F_0,\ M(0) = M_0,\ B_1(0) = B_{10} \tag{4}$$

We should note that system (3) can be reduced to the integer order system (1) in the limit case $\alpha \to 1$.

3 Equilibrium Points And Their Local Asymptotic Stability

Consider the fractional order system (3)-(4) with α satisfying $0 < \alpha \leq 1$.

To evaluate the equilibrium points of (3), let

$$\begin{cases} D^\alpha B = 0, \\ D^\alpha F = 0, \\ D^\alpha M = 0, \\ D^\alpha B_1 = 0. \end{cases}$$

Then the equilibrium points are $E_0 = (0,0,0,0)$ and $E_1 = (B^*, F^*, M^*, B_1^*)$.

The Jacobian matrix $J(E_1)$ for the system given in (3) is

$$J(E_1) = \begin{bmatrix} a-\lambda & x & -x & 0 \\ z & e-\lambda & j & 0 \\ 0 & p & k-\lambda & t \\ 0 & y & -y & a-\lambda \end{bmatrix}$$

where

$$\begin{aligned} x &= bc - 2b(F^* - M^*) \\ y &= -bd + 2b(M^* - F^*) \\ z &= gh - 2gB^* \\ t &= mn - 2mB_1^* \end{aligned}$$

To discuss the local stability of the equilibrium $E_1 = (B^*, F^*, M^*, B_1^*)$ of the system given by (3), we consider the linearized system at E_1. The characteristic equation of the linearized system is of the form

$$p(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0$$

where

$$\begin{aligned}
\alpha_1 &= (-e - 2a - k) \\
\alpha_2 &= (-xz + 2ae + 2ka + ke - pj + a^2) \\
\alpha_3 &= (kxz + pxz + 2paj + axz - a^2e - ka^2 - tyj - ke - kae) \\
\alpha_4 &= (ka^2e - kaxz - pxaz - pa^2j + tajy + txzy)
\end{aligned}$$

According to the fractional Routh–Hurwitz criteria we have the following theorem.

Theorem 1 *If* $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$ *and* $\alpha_1\alpha_2\alpha_3 - \alpha_3^2 - \alpha_1^2\alpha_4 > 0$ *then the equilibrium point* $E_1 = (B^*, F^*, M^*, B_1^*)$ *is locally asymptotically stable for all* $\alpha \in (0, 1]$.

Proof. $E_1 = (B^*, F^*, M^*, B_1^*)$ equilibrium of the system given by (3) is asymptotically stable if all of the eigenvalues, λ_i, $i = 1, 2, 3, 4$ of $J(E_1)$ satisfy the following condition (negative real part) [6,1] :

$$|\arg \lambda_i| > \frac{\alpha\pi}{2}.$$

For $n = 4$, the Routh-Hurwitz criteria are $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$ and $\alpha_1\alpha_2\alpha_3 - \alpha_3^2 - \alpha_1^2\alpha_4 > 0$. It is a necessary and sufficient condition for the negativity of the real parts of all the roots of the polynomial $p(\lambda)$. ■

If one of the conditions above does not hold, model gives rise to unbounded feeling, which is obviously unrealistic.

4 Numerical method

The following theorems are given for solving differential equations of fractional order in [3].

Theorem 2 *Let* $\|.\|$*denote any convenient norm on* R^n. *Assume that* $f \in C[R_1, R^n]$, *where* $R_1 = [(t, X) : 0 \le t \le a$ *and* $\|X - X_0\| \le b]$, $f = (f_1, f_2, ..., f_n)^T$ *and* $X = (x_1, x_2, ..., x_n)^T$ *and let* $\|f(t, X)\| \le M$, *on* R_1. *Then, there exists at least one solution for the system of fractional differential equation given by*

$$D^\alpha X(t) = f(t, X(t)) \quad (5)$$

with the initial condition

$$X(0) = X_0 \tag{6}$$

on $0 \le t \le \beta$ *where* $\beta = \min\left(a, \left[\frac{b}{M}\Gamma(\alpha+1)\right]^{\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$.

Theorem 3 *Consider the IVP given by (5)-(6) of order* α, $(0 < \alpha < 1)$. *Let*

$$g(\nu, X_*(\nu)) = f\left(t - (t^\alpha - \nu\Gamma(\alpha+1))^{1/\alpha}, X\left(t - (t^\alpha - \nu\Gamma(\alpha+1))^{1/\alpha}\right)\right)$$

and assume that the conditions of Theorem 2 hold. Then, a solution $X(t)$ *of (5) can be given by*

$$X(t) = X_*(t^\alpha/\Gamma(\alpha+1))$$

where $X_*(v)$ *is a solution of the system of integer order differential equations*

$$\frac{d(X_*(\nu))}{d\nu} = g(\nu, X_*(\nu))$$

with the initial condition

$$X_*(0) = X_0.$$

5 Numerical solutions and simulations

For the numerical solutions of the given system, we use the technique given by theorem 3. This technique is based on transforming the fractional order system to an integer order system and evaluating the solution of the fractional order system in terms of the solution of the integer order system.

The parameters used in the model are

$a = -1$	$b = -0.1$	$c = 0.0025$	$d = 0$	$u = -0.002$	$g = 0.0003$	$h = 0.002$	$j = 0.001$
$k = 0$	$m = 0.0003$	$n = 0.001$	$p = 0.005$	$\gamma_1 = 1$	$\gamma_2 = 1$	$\gamma_3 = 1$	$\gamma_4 = 5$

with the initial conditions

$$B(0) = 10,\ F(0) = 8,\ M(0) = 6,\ B_1(0) = 4.$$

Positive equilibrium point for the system (3) is calculated as:

$$B^* = 57.6208,\ F^* = 27.78,\ M^* = 51.5739,\ B_1^* = 61.6149$$

The approximate solutions $B(t)$, $F(t)$, $M(t)$, $B_1(t)$ are displayed in figure 1 for $\alpha = 0.95$.

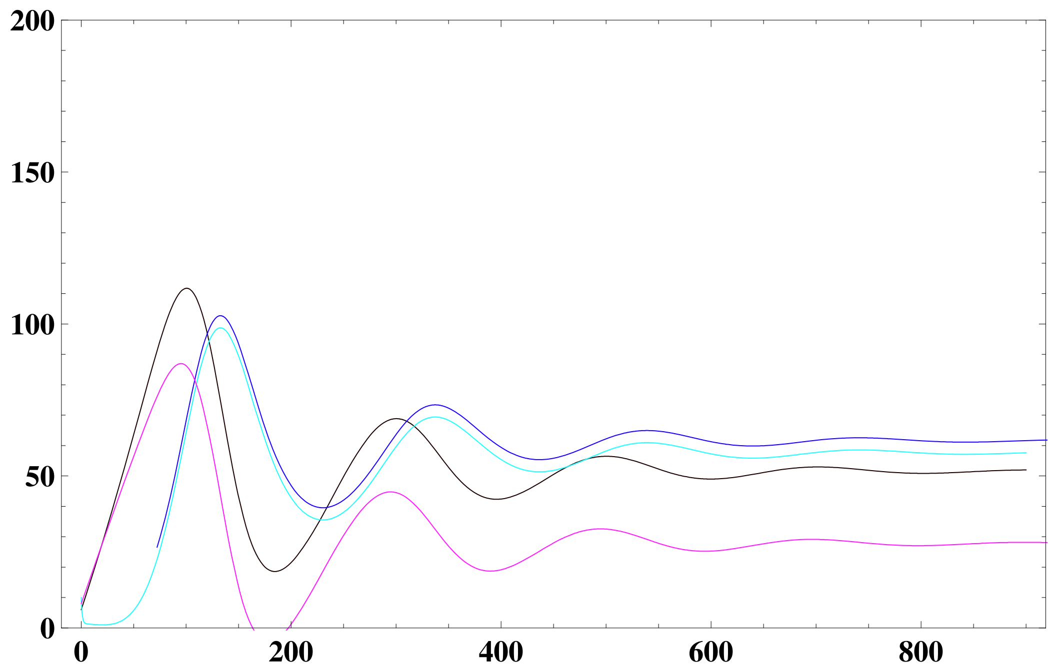

Figure 1: The approximate solutions $B(t)$, $F(t)$, $M(t)$, $B_1(t)$ for $\alpha = 0.95$.

6 Conclusions

In this paper, we give a new integer order mathematical model of nuclear family. Then, we offer a fractional order mathematical model and investigate the stability of solutions. Finally, we give numerical solutions of the given system to illustrate our results.

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