Applied Mathematical Sciences, Vol. 8, 2014, no. 65, 3243 - 3254
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.44256

Numerical Solution of Kawahara Equations by Using Laplace Homotope Perturbations Method

Bothayna S. Kashkari

Department of Mathematics, Sciences Faculty for Girls
King Abdulaziz University, Jeddah, Saudi Arabia

Abstract

In this paper, we applied Laplace Homotope Perturbations Method (LHPM) to solve the Kawahara type equations. The numerical results obtained by (LHPM) are compared with exact solutions, homotopy analysis method (HAM), homotopy perturbation method (HPM), optimal Homotopy Perturbation method (OHPM) and Homotpy Perturbation and Variational Iteration method (VHPM) so that the results reveal the effectiveness of the suggested method and high accuracy.

Keywords: Kawahara equation, Modified Kawahara equation, Laplace Transform, Homotope Perturbations Method.

1 Introduction

In this paper we study numerical solution of Kawahara equation, Modified Kawahara equation respectively given as:

$$u_t + \alpha u u_x + \beta u_{3x} - \gamma u_{5x} = 0 \tag{1.1}$$

$$u_t + u^2 u_x + \beta u_{3x} + \gamma u_{5x} = 0 \tag{1.2}$$

Kawahara equation (1.1) has been derived by Hasimoto [4] as a model for water waves in the long wave regime for moderate values of surface tension, which is nowadays called the Kawahara equation. Historically, this type of equation was first found by Kakutani and Ono [15] in analysis of magnet-acoustic waves in a cold collision free plasma. Then, Hasimoto [4] derived the above equation from capillary-gravity waves. Kawahara [16] studied this type of equation numerically and observed that the equation has both oscillatory and monotone solitary wave

solutions. Modified Kawahara equation (1.2) is known as the critical surface-tension model. This equation arises in the modeling of weakly nonlinear waves in a wide variety of media [15,16].
Recently different analytic and numerical methods [1,3,5,9-13] have been proposed for solving the Kawahara type equations.
The aim of the present work is to effectively employ the (LHPM) to establish approximate solutions for Kawahara type equations The method is tested with the aid of the two numerical examples, for which the exact solution is known. Error estimates show that the accuracy of computations is very high even when the mode number is small. Comparison of the results of present method (LHPM) with (HAM), (HPM), (OHPM), and (VHPM) are also presented in this paper.

2 Laplace Homotope Perturbations Method

In this section, we present a Laplace Homotope Perturbations Method (LHPM) for solving partial differential equations written in an operator form

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \tag{2.1}$$

with the initial conditions

$$u(x,0)=f(x) \tag{2.2}$$

where ***L*** is consider as a first-order partial differential operator, ***R*** is the remaining operator, ***N*** represent a general nonlinear differential operators and ***g(x,t)*** is a source term. According to (LHPM) [7] we apply Laplace transform on both sides of (2.1)

$$\mathfrak{L}\left[Lu(x,t)\right]+\mathfrak{L}\left[Ru(x,t)\right]+\mathfrak{L}\left[Nu(x,t)\right]=\mathfrak{L}\left[g(x,t)\right] \tag{2.3}$$

Using the differential property of Laplace transform and initial conditions we get

$$s\,\mathfrak{L}\left[u(x,t)\right]-u(x,0)+\mathfrak{L}\left[Ru(x,t)\right]+\mathfrak{L}\left[Nu(x,t)\right]=\mathfrak{L}\left[g(x,t)\right] \tag{2.4}$$

$$\mathfrak{L}\left[u(x,t)\right]=\frac{1}{s}u(x,0)-\frac{1}{s}\mathfrak{L}\left[Ru(x,t)\right]-\frac{1}{s}\mathfrak{L}\left[Nu(x,t)\right]+\frac{1}{s}\mathfrak{L}\left[g(x,t)\right] \tag{2.5}$$

Applying inverse Laplace transform to (2.5)

$$u(x,t)=F(x)-\mathfrak{L}^{-1}\left[\frac{1}{s}\left(\mathfrak{L}\left[Ru(x,t)\right]+\mathfrak{L}\left[Nu(x,t)\right]\right)\right] \tag{2.6}$$

where ***F(x)*** represent the terms arising from source term and prescribe initial condition.
By the homotopy technique proposed by Liao [14]

$$u(x,t)=F(x)-p\,\mathfrak{L}^{-1}\left[\frac{1}{s}\left(\mathfrak{L}\left[Ru(x,t)\right]+\mathfrak{L}\left[Nu(x,t)\right]\right)\right] \tag{2.7}$$

the solution of (2.7) can be written as the following form

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t) \tag{2.8}$$

where $p\in[0,1]$ is an embedding parameter. The nonlinear term ***N*** can be written as

$$Nu(x,t)=\sum_{n=0}^{\infty}p^{n}H_{n}(x,t) \tag{2.9}$$

where $H_n(x,t)$ is the He's polynomials [14]

$$H_{n}(u_{0},...,u_{n})=\frac{1}{n!}\frac{\partial^{n}}{\partial p^{n}}\left[N(\sum_{i=0}^{\infty}p^{i}u_{i}(x,t))\right]_{p=0} \tag{2.10}$$

By substituting (2.8) and (2.9) into (2.7) we have

$$\sum_{n=0}^{\infty}p^{n}u_{n}(x,t)=F(x)$$

$$-p\mathfrak{L}^{-1}\left[\frac{1}{s}(\mathfrak{L}\left[R(\sum_{n=0}^{\infty}p^{n}u_{n}(x,t))\right]+\mathfrak{L}\left[N(\sum_{n=0}^{\infty}p^{n}H_{n}(x,t))\right])\right] \tag{2.11}$$

Equating the coefficients of p with the same powers in (2.11) leads to

$$p^{0}:\quad u_{0}(x,t)=F(x)$$

$$p^{1}:\quad u_{1}(x,t)=-\mathfrak{L}^{-1}\left[\frac{1}{s}(\mathfrak{L}\left[Ru_{0}(x,t)\right]+\mathfrak{L}\left[NH_{0}(x,t)\right])\right]$$

$$p^{2}:\quad u_{2}(x,t)=-\mathfrak{L}^{-1}\left[\frac{1}{s}(\mathfrak{L}\left[Ru_{1}(x,t)\right]+\mathfrak{L}\left[NH_{1}(x,t)\right])\right] \tag{2.12}$$

$$\vdots$$

$$p^{n+1}:\quad u_{n+1}(x,t)=-\mathfrak{L}^{-1}\left[\frac{1}{s}(\mathfrak{L}\left[Ru_{n}(x,t)\right]+\mathfrak{L}\left[NH_{n}(x,t)\right])\right]$$

when, $p\to 1$ (2.8) becomes the approximate solution of equation (2.6), i.e.,

$$u(x,t)=\operatorname*{Lim}_{p\to 1}u_{n}(x,t) \tag{2.13}$$

which may converge to the exact solution of (2.1).

3 Numerical results

In this section, we will apply (LHPM) to solve Kawahara equation and the modified Kawahara equation, and present numerical results to verify the effectiveness of the method.

Example 1 : Consider Kawahara equation (1.1) with $\alpha=\beta=\gamma=1$ and with initial condition [2,8,17]

$$u(x,0)=-\frac{72}{169}+\frac{105}{169}\operatorname{sech}^{4}(kx) \tag{3.1}$$

The exact solution of this equation is

$$u(x,t)=-\frac{72}{169}+\frac{105}{169}\operatorname{sech}^{4}(k(x+ct)) \tag{3.2}$$

where $k=\dfrac{1}{2\sqrt{13}}$ and $c=\dfrac{36}{169}$.

Taking Laplace transform of (1.1) we have

$$s\,\mathfrak{L}[u(x,t)] - u(x,0) = -\mathfrak{L}[uu_x] - \mathfrak{L}[u_{3x}] + \mathfrak{L}[u_{5x}] \tag{3.3}$$

By applying the initial condition we obtain

$$\mathfrak{L}[u(x,t)] = \frac{1}{s}\left(-\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\right) - \frac{1}{s}\left[\mathfrak{L}[uu_x] + \mathfrak{L}[u_{3x}] - \mathfrak{L}[u_{5x}]\right] \tag{3.4}$$

Taking inverse Laplace transform of (3.4) we have

$$u(x,t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right) - \mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}[uu_x] + \mathfrak{L}[u_{3x}] - \mathfrak{L}[u_{5x}]\right]\right] \tag{3.5}$$

By substitution of (2.8) and (2.9) in (3.5) we have

$$\begin{aligned}\sum_{n=0}^{\infty} p^n u_n(x,t) &= -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\\ &\quad -p\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\begin{array}{l}\mathfrak{L}\left[\sum_{n=0}^{\infty} p^n H_n(x,t)\right] + \mathfrak{L}\left[\sum_{n=0}^{\infty} p^n (u_n(x,t))_{3x}\right]\\ -\mathfrak{L}\left[\sum_{n=0}^{\infty} p^n (u_n(x,t))_{5x}\right]\end{array}\right]\right]\end{aligned} \tag{3.6}$$

where $H_n(x,t)$ are He's polynomials for the nonlinear term uu_x

$$H_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)_x\right]_{p=0} \tag{3.7}$$

Using primes to indicate the differentiation with respect to ***x***

$$\begin{aligned}
&H_0(u_0) = u_0 u_0'\\
&H_1(u_0,u_1) = u_1 u_0' + u_0 u_1'\\
&H_2(u_0,u_1,u_2) = u_2 u_0' + u_1 u_1' + u_0 u_2'\\
&\vdots\\
&H_n(u_0,\cdots,u_n) = u_n u_0' + u_{n-1} u_1' + \cdots + u_1 u_{n-1}' + u_0 u_n
\end{aligned} \tag{3.8}$$

By equating the coefficients of ***p*** with the same powers in (3.6) leads to

$$u_0(x,t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right) \tag{3.9}$$

$$\begin{aligned}u_1(x,t) &= -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}[H_0(x,t)] + \mathfrak{L}[(u_0(x,t))_{3x}] - \mathfrak{L}[(u_0(x,t))_{5x}]\right]\right]\\ &= -\frac{7560t\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\operatorname{Tanh}\left(\frac{x}{2\sqrt{13}}\right)}{28561\sqrt{13}}\end{aligned} \tag{3.10}$$

$$u_2(x,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_1(x,t)\right] + \mathfrak{L}\left[(u_1(x,t))_{3x}\right] - \mathfrak{L}\left[(u_1(x,t))_{5x}\right]\right]\right]$$

$$= \frac{68040t^2 \operatorname{sech}^6(\frac{x}{2\sqrt{13}})[-3+2\cosh(\frac{x}{\sqrt{13}})]}{62748517} \tag{3.11}$$

$$u_3(x,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_2(x,t)\right] + \mathfrak{L}\left[(u_2(x,t))_{3x}\right] - \mathfrak{L}\left[(u_2(x,t))_{5x}\right]\right]\right]$$

$$= -\frac{816480t^3 \operatorname{sech}^7(\frac{x}{2\sqrt{13}})[2\sinh(\frac{3x}{2\sqrt{13}}) - 13\sinh(\frac{x}{2\sqrt{13}})]}{10604499373\sqrt{13}} \tag{3.12}$$

$$u_4(x,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_3(x,t)\right] + \mathfrak{L}\left[(u_3(x,t))_{3x}\right] - \mathfrak{L}\left[(u_3(x,t))_{5x}\right]\right]\right]$$

$$= \frac{3674160t^4 \operatorname{sech}^8(\frac{x}{2\sqrt{13}})[52+4\cosh(\frac{2x}{\sqrt{13}}) - 49\cosh(\frac{x}{\sqrt{13}})]}{23298085122481} \tag{3.13}$$

$$u_5(x,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_4(x,t)\right] + \mathfrak{L}\left[(u_4(x,t))_{3x}\right] - \mathfrak{L}\left[(u_4(x,t))_{5x}\right]\right]\right]$$

$$= -\frac{13226976t^5 \operatorname{sech}^9(\frac{x}{2\sqrt{13}})}{3937376385699289\sqrt{13}}[8\sinh(\frac{5x}{2\sqrt{13}}) - 171\sinh(\frac{3x}{2\sqrt{13}}) \tag{3.14}$$

$$+661\sinh(\frac{x}{2\sqrt{13}})]$$

And so on, in this manner the rest of components of the series were obtained.
According to (3.9)-(3.14) we obtain

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + u_5(x,t) \tag{3.15}$$

Here we have used five terms to derive the approximate solution.

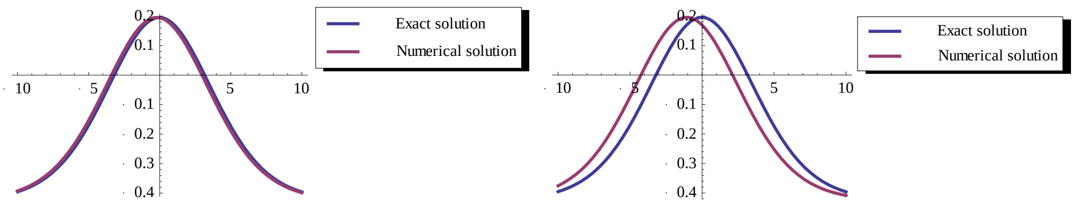
From Table1, we find that (LHAM) has a small error. The results produced by (LHAM) are in a very good agreement with the best of the results of the methods listed in Table2. It is seen in Fig.2 that *u(x,t)* gives a good approximation in the interval $x \in [-10,10]$ at $t = 1, 5$. In Fig.3 the surface shows the exact solution (a) and the approximation solution (b) of Kawahara equation for $-10 \leq x \leq 10$ and $0 \leq t \leq 10$.

Table 1. Absolute error corresponding to example 1 at time $t = 2.0,4.0,6.0,8.0,10.0$ and $1.0 \le x \le 10.0$.

x/t	2	4	6	8	10
1	$3.232454582 \times 10^{-8}$	$1.913685434 \times 10^{-6}$	$1.998530015 \times 10^{-5}$	$1.020029457 \times 10^{-4}$	$3.500405113 \times 10^{-4}$
2	$8.56621149 \times 10^{-9}$	$6.835015515 \times 10^{-7}$	$9.209297425 \times 10^{-6}$	$5.903108491 \times 10^{-5}$	2.5014388×10^{-4}
3	$3.244662007 \times 10^{-8}$	$2.101565772 \times 10^{-6}$	$2.409494832 \times 10^{-5}$	$1.356020223 \times 10^{-4}$	$5.158548843 \times 10^{-4}$
4	$2.910579283 \times 10^{-8}$	$1.807391077 \times 10^{-6}$	$1.991597238 \times 10^{-5}$	$1.079704251 \times 10^{-4}$	$3.965149817 \times 10^{-4}$
5	$1.268792024 \times 10^{-8}$	$7.479598784 \times 10^{-7}$	$7.817374448 \times 10^{-6}$	$4.015494219 \times 10^{-5}$	$1.395478905 \times 10^{-4}$
6	$1.023471585 \times 10^{-9}$	$1.000006379 \times 10^{-7}$	$1.492223182 \times 10^{-6}$	$1.015405687 \times 10^{-5}$	$4.471177813 \times 10^{-5}$
7	$6.679879039 \times 10^{-9}$	$4.335357665 \times 10^{-7}$	$4.983505673 \times 10^{-6}$	$2.813920643 \times 10^{-5}$	$1.074861476 \times 10^{-4}$
8	$6.524592311 \times 10^{-9}$	$4.101178521 \times 10^{-7}$	$4.580296097 \times 10^{-6}$	$2.519669727 \times 10^{-5}$	$9.399544522 \times 10^{-5}$
9	$4.242213358 \times 10^{-9}$	$2.620685313 \times 10^{-7}$	$2.879865573 \times 10^{-6}$	$1.560472552 \times 10^{-5}$	$5.739506365 \times 10^{-5}$
10	$2.055495996 \times 10^{-9}$	$1.247967491 \times 10^{-7}$	$1.348605775 \times 10^{-6}$	$7.190077881 \times 10^{-6}$	$2.603387172 \times 10^{-5}$

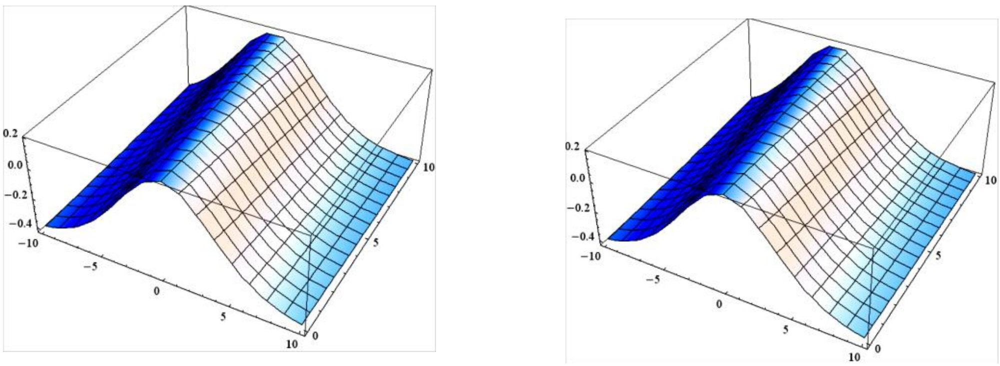
Table 2. Result of the proposed method compared with results in [2,8,17] at time $t = 0.1$ and $0.1 \le x \le 0.5$.

x	OHAM[2]	VHPM[8]	HPM[17]	LHAM
0.1	1.5781×10^{-6}	2.1812×10^{-9}	5.0004×10^{-5}	3.0531×10^{-16}
0.2	2.1006×10^{-6}	4.2408×10^{-9}	1.8880×10^{-4}	4.1633×10^{-16}
0.3	2.6175×10^{-6}	6.2785×10^{-9}	2.1822×10^{-4}	1.1657×10^{-15}
0.4	3.1273×10^{-6}	8.2840×10^{-9}	9.0078×10^{-5}	6.1062×10^{-16}
0.5	3.6289×10^{-6}	1.0247×10^{-8}	1.3129×10^{-3}	8.8818×10^{-16}



(a) $t = 1, -10 \le x \le 10$ (b) $t = 5, -10 \le x \le 10$

Fig.1: The graph of the exact and approximate solution



(a) The surface of the exact solution (b) The surface of the approximation solution

Fig.2: The surface shows the exact and the approximation solution of Kawahara equation for $-10 \le x \le 10$ and $0 \le t \le 10$

Example 2: Consider the modified Kawahara equation (1.2) with $\beta = 0.001, \gamma = -1$ and with initial condition [6-9]

$$u(x,0) = \frac{3\beta}{\sqrt{-10\gamma}} \operatorname{sech}^2(kx) \tag{3.15}$$

The exact solution of this equation is

$$u(x,t) = \frac{3\beta}{\sqrt{-10\gamma}} \operatorname{sech}^2(k(x-ct)) \tag{3.16}$$

where $k = \frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}$ and $c = \frac{25\gamma - 4\beta^2}{25\gamma}$.

Taking Laplace transform of (1.2) we have

$$s\,\mathfrak{L}[u(x,t)] - u(x,0) = -\mathfrak{L}\left[u^2 u_x\right] - \mathfrak{L}[\beta u_{3x}] - \mathfrak{L}[\gamma u_{5x}] \tag{3.17}$$

By applying the initial condition we obtain

$$\begin{aligned} \mathfrak{L}[u(x,t)] &= \frac{1}{s}\left(\frac{3\beta}{\sqrt{-10\gamma}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right)\right) \\ &\quad - \frac{1}{s}\left[\mathfrak{L}\left[u^2 u_x\right] + \mathfrak{L}[\beta u_{3x}] + \mathfrak{L}[\gamma u_{5x}]\right] \end{aligned} \tag{3.18}$$

Taking inverse Laplace transform of (3.18) we have

$$\begin{aligned} u(x,t) &= \frac{3\beta}{\sqrt{-10\gamma}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right) \\ &\quad - \mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[u^2 u_x\right] + \mathfrak{L}[\beta u_{3x}] + \mathfrak{L}[\gamma u_{5x}]\right]\right] \end{aligned} \tag{3.19}$$

By substitution of (2.8) and (2.9) in (3.19) we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x,t) &= \frac{3\beta}{\sqrt{-10\gamma}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right) \\ &\quad - p\,\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\begin{array}{l} \mathfrak{L}\left[\sum_{n=0}^{\infty} p^n H_n(x,t)\right] + \beta\mathfrak{L}\left[\sum_{n=0}^{\infty} p^n (u_n(x,t))_{3x}\right] \\ +\gamma\mathfrak{L}\left[\sum_{n=0}^{\infty} p^n (u_n(x,t))_{5x}\right] \end{array}\right]\right] \end{aligned} \tag{3.20}$$

where $H_n(x,t)$ are He's polynomials for the nonlinear term $u^2 u_x$

$$H_n(u_0,\dots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)^2 \left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)_x\right]_{p=0} \tag{3.21}$$

Using primes to indicate the differentiation with respect to ***x***

$$H_0(u_0) = u_0^2 u_0'$$

$$H_1(u_0,u_1) = 2u_0 u_1 u_0' + u_0^2 u_1'$$

$$H_2(u_0,u_1,u_2) = (u_1^2 + 2u_0u_2)u_0' + 2u_0u_1u_1' + u_0^2u_2'$$
$$\vdots$$
$$\begin{aligned}H_{n(even)}(u_0,\cdots,u_n) &= (u_{\frac{n}{2}}^2 + 2u_{\frac{n}{2}-1}u_{\frac{n}{2}+1} + \cdots + 2u_0u_n)u_0' \\ &+ (2u_{\frac{n}{2}-1}u_{\frac{n}{2}} + 2u_{\frac{n}{2}-2}u_{\frac{n}{2}+1} + \cdots + 2u_0u_{n-1})u_1' \\ &+ \cdots + u_0^2u_n' \end{aligned} \tag{3.22}$$
$$\begin{aligned}H_{n(\mathrm{odd})}(u_0,\cdots,u_n) &= (2u_{\frac{n-1}{2}}u_{\frac{n+1}{2}} + 2u_{\frac{n-1}{2}-1}u_{\frac{n+1}{2}+1} + \cdots + 2u_0u_n)u_0' \\ &+ (u_{\frac{n-1}{2}}^2 + 2u_{\frac{n-1}{2}-1}u_{\frac{n-1}{2}+1} + \cdots + 2u_0u_{n-1})u_1' \\ &+ \cdots + u_0^2u_n'\end{aligned}$$

By equating the coefficients of ***p*** with the same powers in (3.20) leads to

$$u_0(x,t) = \frac{3\beta}{\sqrt{-10\gamma}}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right) \tag{3.23}$$

$$\begin{aligned}u_1(x,t) &= -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_0(x,t)\right] + \beta\mathfrak{L}\left[(u_0(x,t))_{3x}\right] + \gamma\mathfrak{L}\left[(u_0(x,t))_{5x}\right]\right]\right] \\ &= \frac{6\beta^3\sqrt{-\frac{2\beta}{\gamma}}\,t\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right)\mathrm{Tanh}\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right)}{125\sqrt{-\gamma^3}}\end{aligned} \tag{3.24}$$

$$\begin{aligned}u_2(x,t) &= -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_1(x,t)\right] + \beta\mathfrak{L}\left[(u_1(x,t))_{3x}\right] + \gamma\mathfrak{L}\left[(u_1(x,t))_{5x}\right]\right]\right] \\ &= \frac{6\sqrt{\frac{2}{5}}\,\beta^6 t^2\,\mathrm{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right)\left[\cosh\left(\sqrt{\frac{-\beta}{5\gamma}}x\right) - 2\right]}{3125\sqrt{-\gamma^7}}\end{aligned} \tag{3.25}$$

$$\begin{aligned}u_3(x,t) &= -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_2(x,t)\right] + \mathfrak{L}\left[(u_2(x,t))_{3x}\right] + \mathfrak{L}\left[(u_2(x,t))_{5x}\right]\right]\right] \\ &= \frac{4\beta^8 t^3\sqrt{-\frac{2\beta}{\gamma}}\,\mathrm{sech}^5\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right)\left[\sinh\left(\frac{3}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right) - 11\sinh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right)\right]}{390625\sqrt{-\gamma^9}}\end{aligned} \tag{3.26}$$

$$u_4(x,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s}\left[\mathfrak{L}\left[H_3(x,t)\right] + \beta\mathfrak{L}\left[(u_3(x,t))_{3x}\right] + \gamma\mathfrak{L}\left[(u_3(x,t))_{5x}\right]\right]\right]$$
$$= \frac{2\sqrt{\frac{2}{5}}\beta^{11}t^4 \operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\gamma}}x\right)\left[33 - 26\cosh\left(\sqrt{\frac{-\beta}{5\gamma}}x\right) + \cosh\left(2\sqrt{\frac{-\beta}{5\gamma}}x\right)\right]}{9765625\sqrt{-\gamma^{13}}} \tag{3.27}$$

And so on, in this manner the rest of components of the series were obtained.
According to (3.23)-(3.27) we obtain

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) \tag{3.28}$$

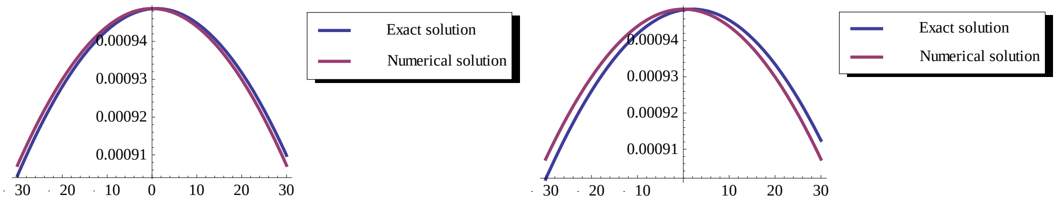
Here we have used four terms to derive the approximate solution.
From Table3, we find that (LHAM) has a small error. The results produced by (LHAM) are in a good agreement with the best of the results of the methods listed in Table 4. It is seen in Fig.3 that *u(x,t)* gives a good approximation in the interval $x \in [-30,30]$ at $t = 1, 2$. In Fig.4 the surface shows the exact solution (a) and the approximation solution (b) of modified Kawahara equation for $-10 \le x \le 10$ and $0 \le t \le 1$.

Table 3. Absolute error corresponding to example 2 at time $t = 2,4,6,8,10$ and $1 \le x \le 10$.

x/t	2	4	6	8	10
1	$6.071174379 \times 10^{-14}$	$3.793471035 \times 10^{-7}$	$1.137434572 \times 10^{-6}$	$2.273051415 \times 10^{-6}$	$3.78438619 \times 10^{-6}$
2	$1.897113036 \times 10^{-7}$	$2.427978608 \times 10^{-13}$	$5.688312321 \times 10^{-7}$	$1.515872474 \times 10^{-6}$	$2.839612082 \times 10^{-6}$
3	$3.793468001 \times 10^{-7}$	$3.79346618 \times 10^{-7}$	$5.461140437 \times 10^{-13}$	$7.58088136 \times 10^{-7}$	$1.893705099 \times 10^{-6}$
4	$5.688306255 \times 10^{-7}$	$7.585418077 \times 10^{-7}$	$5.688301401 \times 10^{-7}$	$9.704160218 \times 10^{-13}$	$9.470423335 \times 10^{-7}$
5	$7.580871053 \times 10^{-7}$	$1.137433784 \times 10^{-6}$	$1.137433481 \times 10^{-6}$	$7.580861956 \times 10^{-7}$	$1.5153665 \times 10^{-12}$
6	$9.47040758 \times 10^{-7}$	$1.515871262 \times 10^{-6}$	$1.705582324 \times 10^{-6}$	$1.515870535 \times 10^{-6}$	$9.470393034 \times 10^{-7}$
7	$1.135616359 \times 10^{-6}$	$1.893703344 \times 10^{-6}$	$2.273049902 \times 10^{-6}$	$2.273049477 \times 10^{-6}$	$1.893702071 \times 10^{-6}$
8	$1.323739004 \times 10^{-6}$	$2.270779642 \times 10^{-6}$	$2.839610026 \times 10^{-6}$	$3.029320966 \times 10^{-6}$	$2.839609056 \times 10^{-6}$
9	$1.511334173 \times 10^{-6}$	$2.646950412 \times 10^{-6}$	$3.405037276 \times 10^{-6}$	$3.784383714 \times 10^{-6}$	$3.784383169 \times 10^{-6}$
10	$1.698327791 \times 10^{-6}$	$3.022066676 \times 10^{-6}$	$3.969107193 \times 10^{-6}$	$4.537937457 \times 10^{-6}$	$4.727648277 \times 10^{-6}$

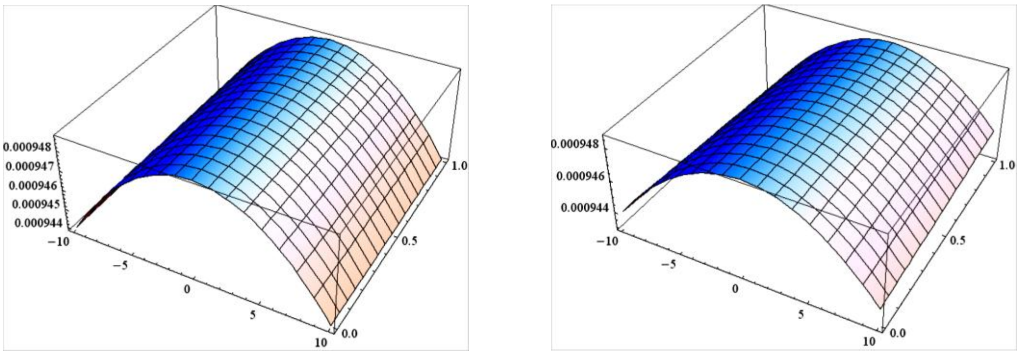
Table 4. Result of the proposed method compared with results in [6-9]

x	*t*	HPM[6]	HAM[9]	LHAM
-5.0	0.02	9.4315×10^{-9}	9.4315×10^{-9}	9.4899×10^{-9}
-2.5	0.04	9.5586×10^{-9}	9.5586×10^{-9}	9.5587×10^{-9}
0.0	0.06	1.7080×10^{-10}	1.7080×10^{-10}	1.7076×10^{-10}
2.5	0.08	1.8663×10^{-8}	1.8663×10^{-8}	1.8663×10^{-8}
5.0	0.1	4.6883×10^{-8}	4.6883×10^{-8}	4.6883×10^{-8}



(a) $t = 1$, $-30 \leq x \leq 30$ (b) $t = 2$, $-30 \leq x \leq 30$

Fig.3: The graph of the exact and approximate solution.



(b) The surface of the exact solution (b) The surface of the approximation solution

Fig.4: The surface shows the exact and the approximation solution of modified Kawahara equation for $-10 \leq x \leq 10$ and $0 \leq t \leq 1$.

4 Conclusion

In this paper, we have successfully used (LHPM) for solving the Kawahara type equations. Numerical results showed that much smaller error with high spatial steps has been achieved by the current method.. They also do not require large computer memory and discretization of the variables $\boldsymbol{t}$ and $\boldsymbol{x}$. The result shows that (LHPM) is powerful mathematical tool for solving nonlinear partial differential equations having wide applications in physical problem represented by Kawahara type equations. MATHEMATICA has been used for computations in this paper.

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Received: April 8, 2014