Applied Mathematical Sciences, Vol. 8, 2014, no. 66, 3255 - 3261
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.45347

A Taylor Series Based Method for Solving a Two-dimensional Second-order Equation

Samaneh Afshar

Department of Mathematics, Islamic Azad University
Central Tehran Branch, Tehran, Iran

Abstract

In this paper, we focus on the two-dimensional linear Telegraph equation with some initial and boundary conditions. We transform the model of partial differential equation (PDE) into a system of first order, linear, ordinary differential equations (ODEs). Our method is based on finding a solution in the form of a polynomial in three variables $U_n(x,y,t) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} U(i,j,k)x^i y^j t^k$ with undetermined coefficients $U(i,j,k)$. The main idea of our process is based on the differential transformation method (DTM).

Keywords: Linear hyperbolic equation; two-dimensional Telegraph equation; Differential Transformation method

1 Introduction

In this paper we focus on the following diffusion equation in two space variables

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha\frac{\partial u}{\partial t} + \beta^2 u = \delta\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^2 u}{\partial y^2} + f(x,y,t),\ 0 < x < 1,\ 0 < y < 1,\ t > 0, \tag{1}$$

while the initial and boundary conditions are

$$u(x,y,0) = p(x,y), \frac{\partial u(x,y,0)}{\partial t} = m(x,y), u(0,y,t) = q(y,t), u(x,0,t) = h(x,t),$$
$$u(1,y,t) = r(y,t), u(x,1,t) = g(x,t),\ 0 \le t \le T,\ 0 \le x \le 1, \tag{2}$$

The functions $f(x,y,t)$, $p(x,y)$, $q(y,t)$, $h(x,t)$, $r(y,t)$, $g(x,t)$ and $m(x,y)$ are known functions and the constants α, β, δ and γ are known constants. For $\alpha > 0$, $\beta = 0$ and $\delta = \gamma = 1$, Eq. (1) represents a damped wave equation and for $\alpha > \beta > 0$ and $\delta = \gamma = 1$, is called telegraph equation. Ding and Zhang[9], presented a three level compact difference scheme of $O(\tau^4 + h^4)$ for the difference solution of problem (1) and (2). An efficient approach for solving the two dimensional linear hyperbolic telegraph equation by the compact finite difference approximation of fourth order and the collocation method have been developed in [18]. Authors of [10, 11, 12] also studied Eq. (1) with other methods. For solving these kinds of equations there are several methods, such as differential transformation method [2, 3, 6, 4, 12, 11, 7, 8, 10, 21, 22, 23, 24, 25], Tau method [19, 20] and homotopy perturbation method [13]. A new matrix formulation technique with arbitrary polynomial bases has been proposed for the numerical/analytical solution of the heat equation with nonlocal boundary condition [1] and Two matrix formulation techniques based on the shifted standard and shifted Chebyshev bases are proposed for the numerical solution of the wave equation with the non-local boundary condition [5].

2 Three-dimensional differential transform

Consider a function of two variables $w(x,y,t)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $w(x,y,t) = \varphi(x)\phi(y)\psi(t)$. Then the function $w(x,y,t)$ can be represented as

$$w(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W(i,j,k)x^i y^j t^k. \tag{3}$$

where $W(i,j,k)$ is called the spectrum of $w(x,y,t)$. Now we introduce the basic definitions and operations of three-dimensional DT as follows[8].

Definition 2.1. Given a w function which has three components such as $x,\ y,\ t$. Three-dimensional differential transform of $w(x,y,t)$ is defined

$$W(i,j,k) = \tfrac{1}{i!j!k!}\left[\tfrac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial t^k} w(x,y,t)\right]_{x=x_0,\ y=y_0, t=t_0}, \tag{4}$$

where the spectrum function $W(i,j,k)$ is the transformed function, which is also called the T-function. let $w(x,y,t)$ as the original function while the uppercase $W(i,j,k)$ stands for the transformed function. Now we define The differential inverse transform of $W(i,j,k)$ as following:

$$w(x,y,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W(i,j,k)(x-x_0)^i(y-y_0)^j(t-t_0)^k. \tag{5}$$

Using Eq. (4) in (5), we have

$$\begin{aligned} w(x,y,t) &= \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i!j!k!}\left[\frac{\partial^{i+j+k}}{\partial x^i\partial y^j\partial t^k}w(x,t)\right]_{x=x_0,\ y=y_0,\ t=t_0}x^iy^jt^k \\ &= \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W(i,j,k)x^iy^jt^k. \end{aligned} \tag{6}$$

Now we give the fundamental theorems for the three-dimensional case of DTM by using the following theorem

Theorem 2.1. *Assume that $W(i,j,k)$, $U(i,j,k)$ and $V(i,j,k)$ are the differential transforms of the functions $w(x,y,t)$, $u(x,y,t)$ and $v(x,y,t)$, respectively; then:*

1− If $w(x,y,t)=u(x,y,t)\pm v(x,y,t)$, then $W(i,j,k)=U(i,j,k)\pm V(i,j,k)$,
2− If $w(x,y,t)=cu(x,y,t)$, where $c\in\mathbf{R}$, then $W(i,j,k)=cU(i,j,k)$
3− If $w(x,y,t)=\frac{\partial}{\partial t}u(x,y,t)$, then $W(i,j,k)=(k+1)U(i,j,k+1)$
4− If $w(x,y,t)=\frac{\partial^{r+s+m}}{\partial x^r\partial y^s\partial t^m}u(x,y,t)$, then

$$W(i,j,k)=\frac{(i+r)!(j+s)!(k+m)!}{i!j!k!}U(i+r,j+s,k+m)$$

Proof. See [8].

3 Reformulation of the problem

In this section, we convert the problem (1) and (2) into a system of first order, linear, ordinary differential equation.
In Eqs. (1) and (2), the functions $f(x,y,t)$, $p(x,y)$, $g(x,y)$, $h(x,t)$, $r(y,t)$ and $m(t)$ generally are not polynomials. We assume that these functions are polynomial or they can be approximated by polynomials to any degree of accuracy. Then if we suppose that

$$\begin{cases} f(x,y,t)\simeq\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}F(i,j,k)x^iy^jt^k, p(x,y)\simeq\sum_{i=0}^{n}\sum_{j=0}^{n}P(i,j)x^iy^j, \\ g(x,y)\simeq\sum_{i=0}^{n}\sum_{j=0}^{n}G(i,j)x^iy^j, h(x,t)\simeq\sum_{i=0}^{n}\sum_{k=0}^{n}H(i,k)x^it^k, \\ r(y,t)\simeq\sum_{j=0}^{n}\sum_{k=0}^{n}R(i,k)y^jt^k, m(x,y)\simeq\sum_{i=0}^{n}\sum_{j=0}^{n}M(i,j)x^iy^j, \end{cases} \tag{7}$$

Therefore we consider approximate solution of the form

$$U_n(x,y,t)=\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}U(i,j,k)x^iy^jt^k, \tag{8}$$

where $U_n(x,y,t)$ is the approximation of $u(x,y,t)$. If we find the values of $U(i,j,k)$, for $i,j,k = 0,1,2,...,n$, then $U_n(x,y,t)$ can be found by using Eq. (7). To find these unknowns, we proceed as follows.
By utilizing Theorem 2.1 and Eqs. (7) ,(8) into Eqs. (1) and (2), we get

$$\begin{cases} (k+1)(k+2)U(i,j,k+2) + \alpha(k+1)U(i,j,k+1) + \beta U(i,j,k) = \delta(i+1)(i+2) \\ U(i+2,j,k) + \gamma(j+1)(j+2)U(i,j+2,k) + F(i,j,k)\ i,j,k = 0,1,2,...,n. \\ \sum_{i=0}^{n}\sum_{j=0}^{n} U(i,j,0)x^i y^j = \sum_{i=0}^{n}\sum_{j=0}^{n} P(i,j)x^i y^j,\ i,j = 0,1,...,n. \\ \sum_{j=0}^{n}\sum_{k=0}^{n} U(0,j,k)y^j t^k = \sum_{j=0}^{n}\sum_{k=0}^{n} Q(j,k)y^j t^k,\ j,k = 0,1,...,n. \\ \sum_{i=0}^{n}\sum_{k=0}^{n} U(i,0,k)x^i t^k = \sum_{i=0}^{n}\sum_{k=0}^{n} H(i,k)x^i t^k,\ i,k = 0,1,...,n. \\ \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} U(i,j,k)y^j t^k = \sum_{j=0}^{n}\sum_{k=0}^{n} R(j,k)y^j t^k,\ j,k = 0,1,...,n. \\ \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} U(i,j,k)x^i t^k = \sum_{i=0}^{n}\sum_{k=0}^{n} G(i,k)x^i t^k,\ i,k = 0,1,...,n. \\ \sum_{i=0}^{n}\sum_{j=0}^{n} U(i,j,1)x^i y^j = M(i,j),\ i,j = 0,1,...,n. \end{cases} \tag{9}$$

Therefore, to finding the values of $U(i,j,k)$, for $i,j,k = 0,1,2,...,n$, we must construct a system of $(n+1)(n+1)(n+1)$ equations. To this end we arrange the obtained linear equations from the previous section. Because the problems (1) and (2) have been not defined for entire of the range, we select only the equations that satisfy in the defined range.
Firstly, the $(n+1)^2$ values of U can be obtained from as follows:

$$\begin{array}{l} U(i,j,0) = P(i,j),\ i,j = 0,1,...,n. \\ U(0,j,k) = Q(j,k),\ j = 0,1,...,n,\ k = 1,2,...,n. \\ U(i,0,k) = H(i,k),\ i,k = 1,2,...,n. \\ U(i,j,1) = M(i,j),\ i,j = 0,1,...,n. \end{array} \tag{10}$$

So, we can obtain $4n^2+5n+2$ values of U independently. Then the n^3-n^2-2n-1 remainder values of U must be obtained from the other equations. We choose $2n^2-9$ equations from he following equations

$$\begin{array}{l} \sum_{i=0}^{n} U(i,j,k) = R(j,k),\ j = 1,...,n-1,\ k = 2,...,n-1, \\ \sum_{i=0}^{n} U(i,1,k) = R(1,k),\ \sum_{i=0}^{n} U(i,n,k) = R(n,k), k = 2,...,n-1, \\ \sum_{i=0}^{n} U(i,j,n) = R(j,n), j = 2,...,n-1, \\ \sum_{j=0}^{n} U(i,j,k) = G(i,k),\ i = 1,...,n-1,\ k = 2,...,n-1, \\ \sum_{j=0}^{n} U(1,j,k) = G(1,k),\ \sum_{j=0}^{n} U(n,j,k) = G(n,k),\ k = 2,...,n-1, \\ \sum_{j=0}^{n} U(i,j,n) = G(i,n),\ i = 2,...,n-2, \end{array} \tag{11}$$

Finally, the remainder n^3-3n^2-2n+8 equations can be found in the form of

$$\begin{aligned}
&(k+1)(k+2)U(i,j,k+2)+\alpha(k+1)U(i,j,k+1)+\beta U(i,j,k)\\
&=\delta(i+1)(i+2)U(i+2,j,k)+\gamma(j+1)(j+2)U(i,j+2,k)\\
&+F(i,j,k), \qquad\qquad i,j=1,2,...,n-2,\ k=0,1,...,n-2,\\
&(k+1)(k+2)U(0,j,k+2)+\alpha(k+1)U(0,j,k+1)+\beta U(0,j,k)\\
&=2\delta U(2,j,k)+\gamma(j+1)(j+2)U(0,j+2,k)\\
&+F(0,j,k), \qquad\qquad j=1,2,...,n-2,\ k=2,...,n-2\\
&(k+1)(k+2)U(i,0,k+2)+\alpha(k+1)U(i,0,k+1)+\beta U(i,0,k)\\
&=\delta(i+1)(i+2)U(i+2,0,k)+2\gamma U(i,2,k)\\
&+F(i,0,k), \qquad\qquad i=1,2,...,n,\ k=2,...,n-2.
\end{aligned} \tag{12}$$

Eqs. (10)-(12) give a linear algebraic system of n^3-n^2-2n-1 equations that by solving this system, the remainder values of U will be obtained and then from Eq. (13), we can obtaine $U_n(x,y,t)$ that it is the approximation of $u(x,y,t)$.

4 Conclusions

In this article, the solution of the second order two space dimensional hyperbolic telegraph equation has been discussed by using differential transformation method. Converting the model of PDE to a system of linear equations, is the main part of this paper. The computational difficulties of the other methods can be reduced by applying this process.

References

[1] B. Soltanalizadeh, Numerical analysis of the one-dimensional Heat equation subject to a boundary integral specification, Optics Communications, 284 (2011), 2109-2112

[2] B. Soltanalizadeh, A. Yildirim, Application of Differential Transformation method for numerical computation of Regularized Long Wave equation, Z. fuer Naturforschung A., 67a (2012), 160-166.

[3] B. Soltanalizadeh, Application of Differential Transformation Method for Numerical Analysis of Kawahara Equation, Australian Journal of Basic and Applied Sciences, 5(12) (2011), 490-495.

[4] B. Soltanalizadeh, Application of Differential Transformation method for solving a fourth-order parabolic partial differential equations, International journal of pure and applied mathematics, 78 (3) (2012), 299-308.

[5] H.R. Ghehsareh, B. Soltanalizadeh,S. Abbasbandy, A matrix formulation to the wave equation with nonlocal boundary condition, Inter. J. Comput. Math., 88 (2011), 1681-1696.

[6] B. Soltanalizadeh, Differential Transformation method for solving one-space-dimensional Telegraph equation, computational applied mathematics, 30(3) (2011), 639-653.

[7] B. Soltanalizadeh, M. Zarebnia, Numerical analysis of the linear and non-linear Kuramoto-Sivashinsky equation by using Differential Transformation method, Inter. J. Appl. Math. Mechanics, 7(12) (2011), 63-72.

[8] F. Ayaz, Solutions of the systems of differential equations by differential transform method, Appl. Math. Comput., 147 (2004), 547-567.

[9] H. Ding, Y. Zhang.A new fourth–order compact finite difference scheme for the two-dimensional second-order hyperbolic equation. J. Comput. Appl. Math., 230 (2009), 626-632.

[10] D. Dengab, C. Zhanga, Application of a fourth-order compact ADI method to solve a two-dimensional linear hyperbolic equation 90 (2013), 273-291.

[11] B. Pekmen, M. Tezer-Sezgin, Differential Quadrature Solution of Hyperbolic Telegraph Equation, doi:10.1155/2012/924765.

[12] A. Mohebbi and M. Dehghan, High order compact solution of the one-space-dimensional linear hyperbolic equation, Numerical Methods for Partial Differential Equations, 24 (2008), 1222-1235.

[13] S.T.Mohyud-Din, A.Yildirim, G.Demirli, Traveling wave solutions of Whitham-Broer-Kaup equations by homotopy perturbation method, Journal of King Saud University-Science, 22 (2010), 173-176 .

[14] A. Shabani Shahrbabaki, R. Abazari, Perturbation method for heat exchange between a gas and solid particles, J. appl. Mech. Tech. Phys, 50(6) (2009), 959-964.

[15] RK. Mohanty, MK. Jain, An unconditionally stable alternating direction implicit scheme for the two-space dimensional linear hyperbolic equation, Numer Meth Partial Diff Eq. 7 (2001), 684-688.

[16] RK. Mohanty, RK. Jainx, U. Arora, An unconditionally stable ADI method for the linear hyperbolic equation in three space dimensional, Int. J. Comput. Math., 79 (2002), 133-142.

[17] M. Dehghan,A. Shokri, A meshless method for numerical solution of a linear hyperbolic equation with variable coefficients in two space dimensions, Numer Meth Partial Diff Eq., 25 (2009), 494-506.

[18] M. Dehghan, A. Mohebbi, High order implicit collocation method for the solution of two-dimensional linear hyperbolic equation, Numer. Meth. Partial Diff. Eq., 25 (2009), 232-243.

[19] B. Soltanalizadeh, H.R. Ghehsareh, S. Abbasbandy, Development of the tau method for the numerical study of a fourth-order parabolic partial differential equation, U.P.B. Sci. Bull., Series A, 75 (2013), 165-176.

[20] B. Soltanalizadeh, H.R. Ghehsareh, S. Abbasbandy, A super accurate Shifted Tau method for numerical computation of the Sobolev-type differential equation with nonlocal boundary conditions, Applied Mathematics and Computation, 236 (2014), 683-692.

[21] B. Babayar, B. Soltanalizadeh, Numerical solution for system of singular nonlinear volterra integro-differential equations by Newton-Product method, Applied mathematics and computation, 219 (2013), 8375-8383.

[22] B. Babayar, B. Soltanalizadeh, Numerical solution of nonlinear singular Volterra integral system by the Newton-Product integration method, Mathematical and computer modelling, 58 (2013), 1696-1703.

[23] B. Soltanalizadeh, HR Ghehsareh, A. Yildirm, S. Abbasbandy, On the Analytic Solution for a Steady Magnetohydrodynamic Equation, Zeitschrift fuer Naturforschung A., 68a (2013), 412-420.

[24] H. Roohani Ghehsareh, S. Abbasbandy, B. Soltanalizadeh, Analytical solutions of the slip MHD viscous flow over a stretching sheet by using the Laplace Adomian Decomposition Method, Zeitschrift fuer Naturforschung A., 67a, (2012), 248-254.

[25] S. Shahmorad, A. Borhanifar, B. Soltanalizadeh, A matrix formulation of the heat equation with nonlocal boundary conditions, Computational Methods in Applied Mathematics, 12 (2012), 92-107.

Received: May 1, 2014