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On the Algebraic Structures of Soft Sets in Logic

Burak Kurt

Department of Mathematics, Faculty of Educations
University of Akdeniz, TR-07058 Antalya, Turkey

Abstract

The main purpose of this study is to introduce and investigate the basic concepts of soft set. After the definitions of soft group and soft ring, we give different soft group and soft rings example.

Keywords: Soft Sets, Soft Sets intersection, Union, Soft Group, Soft Subgroups, Soft homomorphism, Soft epimorphism.

1 Introduction

Researchers in economics, environmental science, engineering, sociology, medical science and any other fields daily with the complexities of modeling uncertain data. Classical methods are not always successful, because the uncertainties appearing in these domain may be of various types. While probability theory, fuzzy set, rough sets and other mathematical tools are well-known and often useful approaches to describing uncertainly, each of these theories has its inherent difficulties as pointed out by Molodtsov [8]. Consequently, Molodtsov [8] proposed a completely new approach for modelling vagueness and uncertainly. This so-called soft set theory is free from the difficulties affecting existing methods.

A soft set is a parameterized family of subsets of the universal set. We can say that soft sets are neighborhood systems. Soft set theory has potential applications in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, probability theory and measurement theory.

Firstly the subject of the soft set is given by Molodtsov [8]. Maji et al. ([6], [7]) defined the binary operation for the soft sets. M. I. Ali et al. [3], Sezgin et al. [10] and Aktas et al. [2] are proved some proposition for the soft groups. Ucar et al. [1] and Feng et al. [5] are defined soft rings and are given some examples for soft rings. Kurt [4] gave an example for soft sets in the modular groups in $\Gamma(n)$.

Definition 1 *Let $a,\ b,\ c,\ d \in \mathbb{C}$. Then following transformations is called Mobius transformation where $ad - bc \neq 0$. These transformation is a non-commutative groups according to matrix multiplication.*

The modular group of order n is defined as

$$\Gamma(n) := \{T :\ ad - bc = n,\ a, b, c, d \in \mathbb{Z}\}.$$

The subgroup of modular groups $\Gamma(n)$ are defined as

$$\Gamma_0(n) := \{T :\ ad - bc = n,\ b \equiv 0 (\operatorname{mod} n),\ a, b, c, d \in \mathbb{Z}\},$$

$$\Gamma^0(n) := \{T :\ ad - bc = n,\ c \equiv 0 (\operatorname{mod} n),\ a, b, c, d \in \mathbb{Z}\}$$

and

$$\Gamma_0^0(n) := \{T :\ ad - bc = n,\ b \equiv c \equiv 0,\ a, b, c, d \in \mathbb{Z}\}.$$

The generating elements of the modular group $\Gamma(1)$ are following:

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The another elements of $\Gamma(1)$ are

$$W = UVU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, P = VU = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

In other words, the subgroups of order two of the modular group $\Gamma(1)$ are respectively $\Gamma_p(2)$, $\Gamma_u(2)$, $\Gamma_v(2)$ and $\Gamma_w(2)$ which is defined as

$$\Gamma_u(2) := \{T \in \Gamma(1) :\ T \equiv I \text{ or } U\ (\operatorname{mod} 2)\},$$

$$\Gamma_v(2) := \{T \in \Gamma(1) :\ T \equiv I \text{ or } V\ (\operatorname{mod} 2)\},$$

$$\Gamma_w(2) := \{T \in \Gamma(1) : \ T \equiv I \text{ or } W \pmod 2\}$$

and

$$\Gamma_p(2) := \left\{T \in \Gamma(1) : \ T \equiv I, \, P \text{ or } P^2 \pmod 2\right\}.$$

These equalities are defined by Rankin [9]. More information can be taken here.

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A \subset E$. Molodtsov [8] defined a soft set in the following manner.

Definition 2 ([7], [8]) *A pair (F, A) is called a soft set over U where F is a mapping given by*

$$F : A \to P(U).$$

In the words, a soft set over U is a parameterized family of subsets of the universe U. For every $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set ε-elements of the soft set (F, A) or as the set of ε-approximate elements of the soft set.

We taken soft set example which is given vintage example by Maji et al. [7].

Example 1 *Suppose the following*

U is the set of houses under consideration,

E is the set of parameters. Each parameter is a word or a sequences.

$$\begin{aligned} E &= \{\textit{expensive; beautiful; wooden; cheap; in the green surroundings;} \\ &\quad \textit{modern; in good repair; in bad repair}\}. \end{aligned}$$

In this case, to define a soft set means to point out expensive houses, beautiful houses and so on. The soft set (F, E) describes the "attractiveness of the houses" which Mr X is going to buy. Suppose that there are six houses in the universe U given by

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\} \ \textit{and } E = \{e_1, e_2, e_3, e_4, e_5\}$$

where

e_1 stands for the parameters "expensive"

e_2 stands for the parameters "beautiful"

e_3 *stands for the parameters "wooden"*

e_4 *stands for the parameters "cheap"*

e_5 *stands for the parameters "the green surroundings".*

Suppose that

$$\begin{aligned} F(e_1) &= \{h_2, h_4\}, \; F(e_2) = \{h_1, h_3\}, \; F(e_3) = \{h_3, h_4, h_5\}, \\ F(e_4) &= \{h_1, h_3, h_5\}, \; F(e_5) = \{h_1\}. \end{aligned}$$

Thus we can view the soft set (F, E) as a collection of approximations as below:

$$\begin{aligned} (F, E) &= \{\textit{expensive house} = \{h_2, h_4\}, \textit{beautiful house} = \{h_1, h_3\}, \textit{wooden house} = \{h_3, h_4, h_5\}, \\ &\quad \textit{cheap house} = \{h_1, h_3, h_5\}, \textit{in the green surroundings house} = \{h_1\}\}. \end{aligned}$$

Definition 3 ([2], [7], [10]) *Let (F, A) and (G, B) be two soft sets over a common universe U. We say that (F, A) is a soft subset of (G, B) If it satisfies:*

i. $A \subset B$,

ii. For $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations

we say that (F, A) is a soft subset of (G, B) denoted by $(F, A) \breve{\subset} (G, B)$.

Definition 4 ([7], [10]) *Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).*

Example 2 *Let $A = \{e_1, e_3, e_5\} \subset E$, $B = \{e_1, e_2, e_3, e_5\} \subset E$. Clearly $A \subset B$. Let (F, A) and (G, B) be two soft sets over the same universe $U = \{h_1, h_2, h_3, h_4, h_5\}$ such that*

$$G(e_1) = \{h_2, h_4\}, \; G(e_2) = \{h_1, h_3\}, \; G(e_3) = \{h_3, h_4, h_5\}, \; G(e_5) = \{h_1\}$$

and

$$F(e_1) = \{h_2, h_4\}, \; F(e_3) = \{h_3, h_4, h_5\}, \; F(e_5) = \{h_1\}.$$

Therefore $(F, A) \breve{\subset} (G, B)$.

Definition 5 (Complement soft set of soft set [3], [7]) *The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \rceil A)$, where $F^c : \rceil A \to P(U)$ is a mapping given by $F^c(\alpha) = U - F(\rceil \alpha)$, $\forall \alpha \in A$.*

Let us call F^c to be the soft complement function of F. Clearly $(F^c)^c$ is the same as F and $((F,A)^c)^c = (F,A)$.

Example 3 (Consider Example 1) *Here*

$$\begin{aligned}(F,E)^c \;=\; & \{\textit{not expensive houses} = \{h_1,h_3,h_5,h_6\}\,,\ \textit{not beautiful houses} = \{h_2,h_4,h_5,h_6\}\,,\\ & \textit{not wooden houses} = \{h_1,h_2,h_6\}\,,\ \textit{not cheap houses} = \{h_2,h_4,h_6\}\\ & \textit{not in the green surroundings houses} = \{h_2,h_3,h_4,h_5,h_6\}\}\,.\end{aligned}$$

Definition 6 (Null Soft Set [7]) *A soft set (F,A) and (G,B) over U is said to be a null soft set denoted by Φ, if $\forall \epsilon \in A$, $F(e) = \varnothing$, null set..*

Definition 7 ([7]) *The union of two soft sets (F,A) and (G,B) over a common universe U is the soft set (H,C) where $C = A \cup B$, $\forall e \in C$,*

$$H(e) = \begin{cases} F(e), & e \in A\backslash B \\ G(e), & e \in B\backslash A \\ F(e) \cup G(e), & e \in A \cap B \end{cases}.$$

We write $(F,A)\,\breve{\cup}\,(G,B) = (H,C)$.

Definition 8 (The Intersection of Two Soft Sets [7], [10]) *(F,A) and (G,B) over a common universe set U is the soft set (H,C), where $C = A \cap B$ and $\forall e \in C$, $H(e) = F(e)$ or $G(e)$ (as both are same set). We write $(F,A)\,\breve{\cap}\,(G,B) = (H,C)$.*

Definition 9 (AND Operation On Two Soft Sets [7], [10]) *If (F,A) and (G,B) are two soft sets over a common universe set U. Then "$(F,A)AND(G,B)$" denoted by $(F,A)\,\breve{\wedge}\,(G,B)$ is defined as $(F,A)\,\breve{\wedge}\,(G,B) = (H,C)$ where $C = A \times B$, $H(x,y) = F(x) \cap G(y)$, for all $(x,y) \in C$.*

Definition 10 (OR Operation On Two Soft Sets [7], [10]) *If (F,A) and (G,B) are two soft sets over a common universe set U. Then "$(F,A)OR(G,B)$" denoted by $(F,A)\,\breve{\vee}\,(G,B)$ is defined by $(F,A)\,\breve{\vee}\,(G,B) = (H,C)$, where $C = A \times B$, $H(x,y) = F(x) \cup G(y)$, for all $(x,y) \in C$.*

2 Soft Group

Throughout this section, G is a group and A is any nonempty set. R will be refer to an arbitrary binary relation between an element of A and an element of G. A set-valued function $F : A \to P(G)$ can be defined as $F(x) = \{y \in G\colon (x,y) \in R,\ x \in A \text{ and } y \in G\}$. The pair (F,A) is then a soft set over G. Defining a set-valued function from A to G also defines a binary relation R on $A \times G$, given by $R = \{(x,y) \in A \times G,\ y \in F(x)\}$. The triplet (A,G,R) is referred to as on approximation set.

Definition 11 ([2]) *Let (F, A) be a soft set cover G. Then (F, A) is said to be a soft group over G if and only if $F(x) < G$ for all $x \in A$.*

Let us illustrate this definition using the following example.

Example 4 *Suppose that D_4-Dihedral group such that*

$$G = A = D_4 = \{e, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$$

and that we define the set-valued function $F(x) = \{y \in G : xRy \Leftrightarrow y = x^n, n \in N\}$. Then the soft group (F, A) is a parameterized family $\{F(x) : x \in A\}$ of subsets, which gives us a collection of subgroups of G. In this case, we can view the soft group (F, A) as the collection of subgroup of G given below:

$$F(e) = \{e\}\,,\ F(1234) = \{e,\ (13)\}\,,\ F((13)(24)) = \{e,\ (13)\}\,,$$

$$F(1432) = \{e,\ (1432)\}\,,\ F((12)(34)) = \{e,\ (12)(34)\}\,,\ F((14)(23)) = \{e,\ (14)(23)\}\,,$$

$$F((13)) = \{e,\ (13)\}\,,\ F((24)) = \{e,\ (24)\}\,.$$

Theorem 1 ([2]) *Let (F, A) and (H, A) be two soft groups over G. Then their intersection $(F, A) \mathbin{\breve{\cap}} (H, A)$ is a soft group over G.*

Proof. From **Definition 8**, we can write $(F, A) \mathbin{\breve{\cap}} (H, A) = (U, C)$ where $C = A \cap A$ and $\forall x \in C$. We have $U(x) = F(x)$ or $U(x) = H(x)$. $U : A \to P(G)$ is a mapping. Therefore (U, A) is a soft set over G. For all $x \in A$, $U(x) = F(x) < G$ or $U(x) = H(x) < G$ since (F, A) and (H, A) are soft groups over G. ■

Theorem 2 *Let (F, A) and (H, B) be two soft groups over G. If $A \cap B = \varnothing$, then $(F, A) \mathbin{\breve{\cup}} (H, B)$ is a soft group over G.*

Proof. From **Definition 7,** we can write $(F, A) \mathbin{\breve{\cup}} (H, B) = (U, C)$. Since $A \cap B = \varnothing$, it follows that either $x \in A \backslash B$ or $x \in B \backslash A$ for all $x \in C$. If $x \in A \backslash B$, then $U(x) = F(x) < G$ and if $x \in B \backslash A$ then $U(x) = H(x) < G$. Thus, $(F, A) \mathbin{\breve{\cup}} (H, B)$ is a soft group over G. ■

Theorem 3 *Let (F, A) and (H, B) be two soft groups over G. Then $(F, A) \wedge (H, B)$ is soft group over G.*

Proof. From **Definition 9,** we can write $(F, A) \wedge (H, B) = (U, A \times B)$. As $F(\alpha)$ and $H(\beta)$ are subgroups of G, $F(\alpha) \cap H(\beta)$ is a subgroup of G. Therefore $U(\alpha, \beta)$ is a subgroup of G for all $(\alpha, \beta) \in A \times B$. Hence we find that $(F, A) \wedge (H, B)$ is a soft group over G. ■

Definition 12 *Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a soft subgroup of (F, A) written $(H, K) \mathrel{\widetilde{<}} (F, A)$ if*

i. $K \subset A$

ii. $H(x) \mathrel{\widetilde{<}} F(x)$ *for all* $x \in K$.

Example 5 *Let D_4 be a Dihedral group G. Let $A = \{\rho_0, \rho_1, \rho_2, \rho_3\}$ be a subgroup of G. In here it is $\rho_0 = e$, $\rho_1 = (1234)$, $\rho_2 = (13)(24)$, $\rho_3 = (1432)$. If we define the function $F(x) = \{y \in G : (x, y) \in R,\ x \in A$ and $y \in B,\ y = x^n,\ n \in \mathbb{N}\}$, $F : A \to P(G)$ then $(H, K) \mathrel{\widetilde{<}} (F, A)$.*

Theorem 4 ([2]) *(F, A) is a soft group over G and $\{(H_i, K_i),\ i \in I\}$ is a nonempty family of soft subgroup of (F, A), where I is an index set . Then*

i. $\bigcap\limits_{i \in I} (H_i, K_i)$ *is a soft subgroup of* (F, A),

ii. $\bigwedge\limits_{i \in I} (H_i, K_i)$ *is a soft subgroup of* (F, A),

iii. If $K_i \cap K_j \neq \varnothing$ for all $i, j \in I$, then $\bigvee\limits_{i \in I} (H_i, K_i)$ is a soft subgroup of (F, A).

Definition 13 ([2]) *Let (F, A) be a soft group over G and (H, B) be soft group of (F, A). Then we say that (H, B) is a normal soft subgroup of (F, A) written $(H, B) \mathrel{\widetilde{\lhd}} (F, A)$, if $H(x)$ is a normal subgroup $F(x)$; i.e. $H(x) \lhd F(x)$, $\forall x \in B$.*

Example 6 *Let $\Gamma(1)$ be a modular group such that $G = A = \Gamma(1)$. Let*

$$F(x) = \{y \in G : xRy \Leftrightarrow y = x^n, n \in \mathbb{N}\}$$

and $B = \Gamma_p(2)$. (F, A) is soft group on G and (H, B) is subgroup of (F, A). For $\forall x \in B$, it is $(H, \Gamma_p(2)) \mathrel{\widetilde{<}} (F, \Gamma(1))$ from $H(x) \mathrel{\widetilde{\lhd}} F(x)$.

3 Soft Rings

From now on $\mathbb{R}$ denotes a commutative ring and all soft sets are considered over $\mathbb{R}$.

Definition 14 *Let (F, A) be a soft. The set $\sup(F, A) = \{x \in A;\ F(x) \neq \varnothing\}$ is called the support of the soft set (F, A). A soft set is said to be non-null if its support is not equal to the empty set.*

Definition 15 ([2]) *Let (F, A) be a non-null soft set over a ring R. Then (F, A) is called a soft ring over R if $F(x)$ is a subring of R for all $x \in A$.*

Example 7 *Let $R = A = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Consider the set-valued function $F : A \to G(R)$ given by $F(x) = \{y \in R,\ xy = 0\}$. Then $F(0) = R$, $F(1) = \{0\}$, $F(2) = \{0, 3\}$, $F(3) = \{0, 2, 4\}$, $F(4) = \{0, 3\}$ and $F(5) = \{0\}$. As we see, all of these sets are subrings of R. Hence (F, A) is a soft ring over R.*

Theorem 5 *Let (F, A) and (G, B) be soft ring over R. Then $(F, A) \breve{\wedge} (G, B)$ is a soft ring over R if it is non-null.*

Proof. By **Definition 9**. Let $(F, A) \breve{\wedge} (G, B) = (H, C)$, where $C = A \times B$ and $H(a, b) = F(a) \cap G(b)$, for all $(a, b) \in C$. Since (H, C) is non-null, $H(a, b) = F(a) \cap G(b) \neq \varnothing$. Since the intersection of any number of subrings of R is a subring of R. $H(a, b)$ is a subring of R. Hence (H, C) is a soft ring over R. ■

Definition 16 *Let (F, A) and (G, B) be soft ring over R. Then (G, B) is called a soft subring of (F, A) if it satisfies the following*

$$i.\ B \subset A$$

$$ii.\ G(x) \text{ is a subring of } F(x), \text{ for all } x \in \sup(G, B).$$

Example 8 *Let $R = A = 2\mathbb{Z}$ and $B = 6\mathbb{Z} \subset A$. Consider the set-valued functions $F : A \to P(R)$ and $G : B \to P(R)$ given by $F(x) = \{nx : n \in \mathbb{Z}\}$ and $G(x) = \{5nx : n \in \mathbb{Z}\}$. As we see, for all $x \in B$, $G(x) = 5x\mathbb{Z}$ is a subring of $x\mathbb{Z} = F(x)$. Hence (G, B) is a soft subring of (F, A).*

Theorem 6 *Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft rings over R. Then;*

$$i.\ \breve{\bigwedge_{i \in I}} (F_i, A_i) \text{ is a soft ring over if it is non-null}$$

$$ii.\ \text{If } \{A_i\colon i \in I\} \text{ are pair wise disjoint, then } \breve{\bigcup_{i \in I}} (F_i, A_i) \text{ is a soft ring over } R.$$

Proof. Clear. ■

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