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Some Properties of Special Polynomials

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Abstract

In this paper, we study some special numbers and polynomials which are closely related to Changhee numbers and polynomials. From our special numbers and polynomials, we derive some new interesting identities.

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1. Introduction

As is well known, the Euler numbers are defined by the generating function to be

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad (\text{see [1–11]})\,. \tag{1.1}$$

Also, Euler polynomials are given by

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l, \quad (n \geq 0). \tag{1.2}$$

It is known that the Changhee polynomials are given by the generating function to be

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_n(x) \frac{t^n}{n!}, \quad (\text{see } [5]). \tag{1.3}$$

When $x = 0$, $\mathrm{Ch}_n = \mathrm{Ch}_n(0)$ are called the Changhee numbers.

By (1.1), (1.2) and (1.3), we easily get

$$\mathrm{Ch}_n(x) = \sum_{l=0}^{n} S_1(n,l) E_l(x), \quad (\text{see } [1\text{–}7]), \tag{1.4}$$

where $S_1(n,l)$ is the Stirling number of the first kind.

From (1.3), we can derive the following recurrence relation:

$$2\,\mathrm{Ch}_n + n\,\mathrm{Ch}_{n-1} = \begin{cases} 2, & \text{if } n = 0 \\ 0, & \text{if } n \geq 1. \end{cases} \tag{1.5}$$

Assume that $p \in \mathbb{Z}$ with $p \geq 2$. Recently, Korobov defined the following special numbers and polynomials ([9]):

$$P_0 = 1, \quad \binom{p}{1} P_n + \cdots + \binom{p}{n+1} P_0 = 0, \quad (n \geq 1), \tag{1.6}$$

and

$$P_0(x) = 1, \quad P_n(x) = \binom{x}{n} P_0 + \cdots + \binom{x}{1} P_{n-1} + P_n, \quad (n \geq 1). \tag{1.7}$$

In this paper, we study some special numbers and polynomials which are different from those of Korobov. Indeed, these numbers and polynomials are closely related to Changhee numbers and polynomials. From our special numbers and polynomials, we derive some new interesting identities.

2. Special numbers and polynomials

Let us consider the following special numbers C_n and special polynomials $C_n(x)$:

$$C_0 = 1, \quad C_0 \binom{p}{n+1} + C_1 \binom{p}{n} + \cdots + C_n \binom{p}{1} + 2C_{n+1} = 0, \quad (n \geq 0), \tag{2.1}$$

and

$$C_0(x) = 1, \quad C_n(x) = C_0 \binom{x}{n} + \cdots + C_{n-1} \binom{x}{1} + C_n, \quad (n \geq 1). \tag{2.2}$$

Then we obtain the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$ and $\Delta f(x) = f(x+1) - f(x)$. Then we have*

$$C_n(0) = C_n, \quad \text{and } \Delta C_n(x) = C_{n-1}(x).$$

Proof. It is easy to show that

$$\Delta \binom{x}{n} = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1}.$$

Thus, we have

$$\Delta C_n(x) = \sum_{l=0}^{n} C_l \Delta \binom{x}{n-l} = \sum_{l=0}^{n} C_l \binom{x}{n-1-l} = C_{n-1}(x).$$

Also, $C_n(0) = C_n$, from (2.1) and (2.2). □

Now, we observe that

$$\begin{aligned} 2 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} C_l \binom{p}{n-l} + C_n \right) t^n \qquad (2.3) \\ &= \left(\sum_{l=0}^{\infty} C_l t^l \right) \left(\sum_{m=0}^{\infty} \binom{p}{m} t^m + 1 \right) \\ &= \left(\sum_{l=0}^{\infty} C_l t^l \right) \left((1+t)^p + 1 \right). \end{aligned}$$

From (1.3) and (2.3), we note that

$$\frac{2}{(1+t)^p + 1} = \sum_{l=0}^{\infty} C_l t^l, \qquad (2.4)$$

and $C_n = \mathrm{Ch}_n$ are the Changhee numbers for $p = 1$. Now, we obtain the following theorem.

Theorem 2.2. *For $p \in \mathbb{Z}$ with $p \geq 2$, we have*

$$\frac{2}{(1+t)^p + 1} = \sum_{l=0}^{\infty} C_l t^l.$$

By (2.2), we easily get

$$\begin{aligned} \sum_{n=0}^{\infty} C_n(x) t^n &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} C_{n-l} \binom{x}{l} \right) t^n \qquad (2.5) \\ &= \left(\sum_{m=0}^{\infty} C_m t^m \right) \left(\sum_{l=0}^{\infty} \binom{x}{l} t^l \right) \\ &= \frac{2}{(1+t)^p + 1} (1+t)^x, \end{aligned}$$

and note that $C_n(x) = \mathrm{Ch}_n(x)$ are just the Changhee polynomials for $p = 1$ (see (1.3)). Therefore, by (2.5), we obtain the following theorem.

Theorem 2.3. *For $p \in \mathbb{Z}$ with $p \geq 2$, we have*

$$\sum_{n=0}^{\infty} C_n(x) t^n = \frac{2}{(1+t)^p + 1}(1+t)^x .$$

From Theorem 2.3, we note that

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(C_n(x+p) + C_n(x)\right) \frac{t^n}{n!} \\ =&\frac{2}{(1+t)^p+1}(1+t)^{x+p} + \frac{2}{(1+t)^p+1}(1+t)^x \\ =&2(1+t)^x = \sum_{n=0}^{\infty} 2\binom{x}{n} t^n. \end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$C_n(x+p) + C_n(x) = 2\binom{x}{n}.$$

For $k \in \mathbb{N}$, we note that

$$\begin{aligned} \sum_{l=0}^{k-1} e^{lt} = \frac{e^{kt}-1}{e^t-1} &= \frac{1}{t}\left(\frac{t}{e^t-1}e^{kt} - \frac{t}{e^t-1}\right) \\ &= \sum_{n=0}^{\infty}\left\{\frac{B_{n+1}(k) - B_{n+1}}{n+1}\right\}\frac{t^n}{n!}. \end{aligned}$$

Thus, we have

$$\sum_{l=0}^{k-1} l^n = \frac{B_{n+1}(k) - B_{n+1}}{n+1}. \tag{2.7}$$

For $n \in \mathbb{N}$ and $x \in \mathbb{N}$, by (2.7), we get

$$\sum_{z=1}^{x-1} z^n = \frac{B_{n+1}(x) - B_{n+1}}{n+1}. \tag{2.8}$$

For $\nu = 0, 1, 2, \ldots, n$, Morgan numbers $M_\nu(n)$ are defined by

$$x^n = \sum_{\nu=0}^{n} \binom{x}{\nu} M_\nu(n). \tag{2.9}$$

From (2.8) and (2.9), we can derive the following equation:

$$\begin{aligned} &\frac{B_{n+1}(x) - B_{n+1}}{n+1} \\ =&\sum_{z=1}^{x-1} z^n \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&=\sum_{z=1}^{x-1}\left(\sum_{\nu=1}^{n}M_{\nu}\left(n\right)\binom{z}{\nu}\right)\\
&=\sum_{\nu=1}^{n}M_{\nu}\left(n\right)\left(\sum_{z=1}^{x-1}\binom{z}{\nu}\right)\\
&=\sum_{\nu=1}^{n}M_{\nu}\left(n\right)\sum_{z=1}^{x-1}\left(\binom{z+1}{\nu+1}-\binom{z}{\nu+1}\right)\\
&=\sum_{\nu=1}^{n}M_{\nu}\left(n\right)\binom{x}{\nu+1}.
\end{aligned}
$$

Therefore, by Theorem 2.4 and (2.10), we obtain the following theorem.

Theorem 2.5. *For $n, x \in \mathbb{N}$, we have*

$$\frac{B_{n+1}\left(x\right)-B_{n+1}}{n+1}=\sum_{\nu=1}^{n}M_{\nu}\left(n\right)\left(\frac{C_{\nu+1}\left(x+p\right)+C_{\nu+1}\left(x\right)}{2}\right).$$

Remark. Note that

$$E_{n}\left(x\right)=\sum_{l=0}^{n}\binom{n}{l}x^{l}E_{n-l}. \tag{2.11}$$

Thus, by (2.11), we get

$$
\begin{aligned}
E_{n}\left(x\right)&=\sum_{l=0}^{n}\binom{n}{l}E_{n-l}\sum_{\nu=0}^{l}\binom{x}{\nu}M_{\nu}\left(l\right)\\
&=\sum_{\nu=0}^{n}\binom{x}{\nu}\left(\sum_{l=\nu}^{n}\binom{n}{l}E_{n-l}M_{\nu}\left(l\right)\right).
\end{aligned}
$$

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