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Bayesian Estimation of Rayleigh Distribution Based on Generalized Order Statistics

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Abstract

Exact expressions for single and product moments of generalized order statistics are derived when the underlying population is assumed to have a Rayleigh distribution. Bayesian estimators and highest posterior density (HPD) credible intervals are obtained for the scale parameter and reliability function based on generalized order statistics. We also derive the Bayes predictive estimator and HPD prediction interval for an independent future observations.

Keywords : Bayes estimation, Generalized order statistics, HPD credible interval

1 Introduction

Let F be an absolutely continuous distribution function with density function f. Let n be a natural number and $k > 0, m_1, \cdots, m_{n-1} \in R$ be given constants such that $\gamma_i = k + n - i + M_i > 0$ for all $1 \leq i \leq n-1$, where $M_i = \sum_{j=i}^{n-1} m_j$. Suppose $Y(1, n, m, k), \cdots, Y(n, n, m, k)$ are the first n of the generalized order statistics based on F. The concept of generalized order statistics is introduced by Kamps (1995). The joint probability density function $f_{1,\cdots,n}(y_1,\cdots,y_n)$ is given by

$$f_{1,\cdots,n}(y_1,\cdots,y_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(y_i)]^{m_i}f(y_i)\right)\left[\bar{F}(y_n)\right]^{k-1}f(y_n), \quad (1.1)$$

where $F^{-1}(0) \leq y_1 < \cdots < y_n < F^{-1}(1)$.

If $m_i = 0$ and $k = 1$, then $Y(r,n,m,k)$ reduces to the ordinary rth order statistic and (1.1) is the joint probability density function of n order statistics $Y_1 \leq \cdots \leq Y_n$. If $m_i = -1$ and $k = 1$ then (1.1) is the joint probability density function of the first n upper record value of a sequence $\{Y_n, n \geq 1\}$ of independent and identically distributed random variables with distribution function F. Also, if $m_i = R_i$, $1 \leq i \leq n-1$, $k = R_n + 1$ and R_i is natural number, (1.1) is the joint probability density function of $(Y_1^{R_1},\cdots,Y_n^{R_n})$ based on progressive type II censored sample with censoring scheme $(R_1,\cdots,R_n)$. Therefore, the notion of generalized order statistics provides a unified approach to some distributional and moment properties of ordered random variables. The results obtained in this paper can be generalized to the Rayleigh distribution based on progressive type II censored sample in Wu et al. (2006). Kim and Han (2009) considered the estimation of the scale parameter of Rayleigh distribution under general progressive censoring.

For various distributional properties and estimation of generalized order statistics, Ashanullah (1996, 2000), Habibullah and Ahsanullah (2000) and Malinowsska et al. (2006) considered the minimum variance unbiased estimator of location and scale parameters of uniform, exponential, Pareto II and Burr XII distributions, respectively. Recently, Abushal and Al-Zaydi (2012) considered the prediction of mixture of Rayleigh distribution using MCMC algorithm.

Let Y be a random variable whose probability density function(p.d.f) is given by

$$g(y) = \frac{y}{\sigma^2}\exp\left(-\frac{y^2}{2\sigma^2}\right), \quad (1.2)$$

and its corresponding cumulative distribution function(c.d.f) is

$$G(y) = 1 - \exp\left(-\frac{y^2}{2\sigma^2}\right). \quad (1.3)$$

The Rayleigh distribution is widely used in communication engineering and life testing of electro vacum devices and is a special case of the two parameter Weibull distribution.

Let $X = Y/\sigma$. Then the random variable X has a standard form of the Rayleigh distribution with c.d.f and p.d.f are written by respectively

$$F(x) = 1 - \exp\left(-\frac{x^2}{2}\right),$$
$$f(x) = x\bar{F}(x), \tag{1.4}$$

where $\bar{F}(x) = 1 - F(x)$.

In this paper, our main object is to describe the variance, covariance and Bayes estimation procedures for the scale parameter σ of the Rayleigh distribution based on generalized order statistics assuming the natural conjugate prior. The rest of this paper is organized as follows. In Section 2, a brief description of generalized order statistics and some distributional properties of the generalized order statistics from Rayleigh distribution are considered. In Section 3, we suggest Bayes estimators and highest posterior density (HPD) credible intervals for the scale parameter and the reliability function of Rayleigh distribution using generalized order statistics. Also, we consider the Bayesian two sample prediction problems.

2 Variance and Covariance of Generalized Order Statistics

Let $Y(1, n, m, k), \cdots, Y(n, n, m, k)$, ($k \geq 1$, m is a real number), be the first n of the generalized order statistics from $\{Y_n, n \geq 1\}$. Then $X(r, n, m, k) = Y(r, n, m, k)/\sigma$, $r = 1, \cdots, n$ are the sequence of generalized order statistics from $\{X_n, n \geq 1\}$. Let

$$v_{rr} = Var[X(r, n, m, k)],$$
$$e_{rs} = E[X(r, n, m, k)X(s, n, m, k)],$$
$$v_{rs} = Cov[X(r, n, m, k), X(s, n, m, k)], \quad r, s = 1, \cdots, n, \ r < s.$$

Reverting to the observations, we have

$$Var[Y(r, n, m, k)] = \sigma^2 v_{rr}, \tag{2.1}$$
$$Cov[Y(r, n, m, k), X(s, n, m, k)] = \sigma v_{rs}. \tag{2.2}$$

We consider the variance and covariance of generalized order statistics from the c.d.f. $F(x)$. Let $X_r \equiv X(r, n, m, k)$ and $X_s \equiv X(s, n, m, k)$ be the rth and sth generalized order statistics from c.d.f. (1.4), respectively.

The marginal density function of the rth generalized order statistics when $m_1 = \cdots = m_{n-1} = m$ and X_r based on the distribution function F and density function f is obtained by integrating out $x_1, \cdots, x_{r-1}$ and $x_{r+1}, \cdots, x_n$ from (1.1) as (for details, refer to Kamps (1995))

$$f_{r,n,m,k}(x) = \frac{c_r}{\Gamma(r)}[\bar{F}(x)]^{\gamma_r - 1} f_\theta(x) g_m^{r-1}(F_\theta(x)), \tag{2.3}$$

where $c_r = \prod_{j=1}^{r} \gamma_j$, $\gamma_j = k + (n-j)(m+1)$ and

$$\begin{aligned} g_m(x) &= \frac{1}{m+1}\left(1-(1-x)^{m+1}\right), \quad m \neq -1, \\ &= -ln(1-x), \quad m=-1. \end{aligned}$$

By substituting $\bar{F}(x)$ and $f(x)$ from (1.4) in (2.3), the marginal density function $f_{r,n,m,k}(x)$ is given by

$$\begin{aligned} f_{r,n,m,k}(x) &= \frac{c_r}{\Gamma(r)}\left(\frac{1}{m+1}\right)^{r-1} x \\ &\times \exp\left(-\frac{\gamma_r x^2}{2}\right)\left[1-\exp\left(-\frac{(m+1)x^2}{2}\right)\right]^{r-1}. \end{aligned} \tag{2.4}$$

From (2.4), the qth moment of the rth generalized order statistics is given by

$$\begin{aligned} E[X(r,n,m,k)^q] &= \frac{c_r}{\Gamma(r)}\left(\frac{1}{m+1}\right)^{r-1}\int_0^\infty x^{q+1}\exp\left(-\frac{\gamma_r x^2}{2}\right) \\ &\times \left(1-\exp\left(-\frac{(m+1)x^2}{2}\right)\right)^{r-1} dx \\ &= \frac{c_r}{\Gamma(r)}\left(\frac{1}{m+1}\right)^{r-1}\sum_{l=0}^{r-1}(-1)^l \binom{r-1}{l} \frac{\Gamma\left(\frac{q+2}{2}\right)}{2\left[\frac{\gamma_{r-l}}{2}\right]^{\frac{q+2}{2}}}. \end{aligned}$$

Therefore, the first and second moments are obtained by letting $q=1$ and $q=2$, respectively, as follows:

$$E[X(r,n,m,k)] = \frac{c_r}{\Gamma(r)}\left(\frac{1}{m+1}\right)^{r-1}\sqrt{\frac{\pi}{2}}\sum_{l=0}^{r-1}(-1)^l \binom{r-1}{l}\frac{1}{[\gamma_{r-l})]^{\frac{3}{2}}}, \tag{2.5}$$

and

$$E[X(r,n,m,k)^2] = \frac{c_r}{\Gamma(r)}\left(\frac{1}{m+1}\right)^{r-1}\sum_{l=0}^{r-1}(-1)^l \binom{r-1}{l}\frac{2}{[\gamma_{r-l}]^2}. \tag{2.6}$$

Lemma 2.1 For $1 \leq r < s \leq n$, let X_r and X_s be the rth and sth generalized order statistics from the c.d.f in (1.4). Then

$$\begin{aligned} E[X(r,n,m,k)X(s,n,m,k)] &= \frac{c_s}{\Gamma(r)\Gamma(s-r)}\left(\frac{1}{m+1}\right)^{r-1} \\ &\times \sum_{l_1=0}^{r-1}\sum_{l_2=0}^{s-r-1}(-1)^{l_1+l_2}\binom{r-1}{l_1}\binom{s-r-1}{l_2} \\ &\times \frac{2\left(-b_{l_2}^2+(a_{l_1,l_2}+b_{l_2})^2_2F_1\left[\frac{1}{2},2,\frac{3}{2};-\frac{a_{l_1,l_2}}{b_{l_2}}\right]\right)}{a_{l_1,l_2}b_{l_2}^2(a_{l_1,l_2}+b_{l_2})^2}, \end{aligned} \tag{2.7}$$

where $a_{l_1,l_2} = (m+1)(s-r+l_1-l_2)$, $b_{l_2} = \gamma_{s-l_2}$ and ${}_2F_1$ denotes the hypergeometric function.

Proof To find the covariance between $X(r,n,m,k)$ and $X(s,n,m,k)$, $r < s$, we consider the joint probability density function $f_{r,s}(u,v)$ of $X(r,n,m,k)$ and $X(s,n,m,k)$. The joint p.d.f. $f_{r,s}(u,v)$ of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by

$$\begin{aligned} f_{r,s}(u,v) &= \frac{c_s}{\Gamma(r)\Gamma(s-r)}\left(\bar{F}(u)\right)^m (g_m(F(u)))^{r-1}\left(\bar{F}(v)\right)^{\gamma_s-1} f(u)f(v) \\ &\times \left[g_m(F(v))-g_m(F(u))\right]^{s-r-1}, \end{aligned} \tag{2.8}$$

where $F^{-1}(0) < u < v < F^{-1}(1)$.

By substituting $\bar{F}(x)$ and $f(x)$ from (1.4) in (2.8) and after some algebraic computations, the product moment of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by

$$\begin{aligned} E[UV] &= \int_0^\infty\int_0^v uvf_{r,s}(u,v)dudv \\ &= \frac{c_s}{\Gamma(r)\Gamma(s-r)}\left(\frac{1}{m+1}\right)^{r-1}\sum_{l_1=0}^{r-1}\sum_{l_2=0}^{s-r-1}(-1)^{l_1+l_2}\binom{r-1}{l_1} \\ &\times \binom{s-r-1}{l_2}\int_0^\infty\left[-\frac{v}{a}\exp\left(-\frac{av^2}{2}\right)+\frac{1}{a^{\frac{3}{2}}}\sqrt{\frac{\pi}{2}}erf\left(\sqrt{\frac{a}{2}}v\right)\right] \\ &= \frac{c_s}{\Gamma(r)\Gamma(s-r)}\left(\frac{1}{m+1}\right)^{r-1}\sum_{l_1=0}^{r-1}\sum_{l_2=0}^{s-r-1}(-1)^{l_1+l_2}\binom{r-1}{l_1} \\ &\times \binom{s-r-1}{l_2}\left(-\frac{2}{a_{l_1,l_2}(a_{l_1,l_2}+b_{l_2})^2}\right. \\ &\left.+\frac{2}{a_{l_1,l_2}b_{l_2}^2}\,{}_2F_1\left[\frac{1}{2},2,\frac{3}{2};-\frac{a_{l_1,l_2}}{b_{l_2}}\right]\right), \end{aligned}$$

where $erf(z) = 2/\sqrt{\pi}\int_0^z e^{-t^2}dt$ which is the integral of the Gaussian distribution and ${}_2F_1$ denotes a hypergeometric function (see Abramowitz and Stegun (1970)). Hence the product moment between $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by (2.7).

According to Lemma 2.1, (2.1) and (2.2), the variance of rth generalized order statistics and covariance between $Y(r,n,m,k)$ and $Y(s,n,m,k)$, $r < s$ are obtained by respectively

$$Var(Y(r,n,m,k)) = \sigma^2\left(b_{r,m}\left[2a_{r,2} - \frac{\pi}{2}b_{r,m}\left(a_{r,\frac{3}{2}}\right)^2\right]\right),$$

and

$$Cov(Y(r,n,m,k), Y(s,n,m,k)) = \sigma^2\left(e_{rs} - 4b_{s,m}b_{r,m}a_{s,2}a_{r,2}\right),$$

where

$$\begin{aligned} b_{i,m} &= \frac{c_i}{\Gamma(i)}\left(\frac{1}{m+1}\right)^{i-1}, \quad i = r, s, \\ a_{i,j} &= \sum_{l=0}^{i-1}(-1)^l\begin{pmatrix} i-1 \\ l \end{pmatrix}\frac{1}{[\gamma_{i-l}]^j}, \quad i = r, s; \quad j = 2, \ \frac{3}{2}, \end{aligned}$$

and e_{rs} is given by (2.7).

Remark If we take $m = 0$ and $k = 1$ then $X(r,n,m,k)$ coincides with the rth order statistics from the sample of size n. In this case, $\gamma_{i-l} = n - i + l + 1$, and $b_{i,0} = c_i/\Gamma(i)$, $i = r, \ s$.

3 Bayes Estimation

3.1 Bayes estimators

Let $\tilde{m} = (m_1, \cdots, m_{n-1})$ and $\mathbf{Y} = (Y(1,n,\tilde{m},k), \cdots, Y(n,n,\tilde{m},k))$ be the first n of the generalized order statistics of $\{Y_n, n \geq 1\}$ from the Rayleigh distribution given by (1.2). Then by substituting $G(y)$ and $g(y)$ from (1.2) and (1.3) in (1.1), the likelihood function $L(\theta; \mathbf{Y})$ is proportional to

$$L(\sigma; \mathbf{Y}) \propto \left(\frac{1}{\sigma}\right)^{2n}\exp\left(-\frac{\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2}{2\sigma^2}\right). \tag{3.1}$$

It is easy to obtain the maximum likelihood estimator (MLE) of σ which is

$$\hat{\sigma}_M = \sqrt{\frac{\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2}{2n}}.$$

By the invariance property of the MLE, the MLE of the reliability function $R(t)$ with fixed $t > 0$ is given by

$$\hat{R}_M = \exp\left(-\frac{t^2}{2\hat{\sigma}_M}\right).$$

Since the σ is a random variable, we consider the natural conjugate family of prior distributions for the σ used in Fernández (2000). It is

$$\pi(\sigma) \propto \left(\frac{1}{\sigma}\right)^{2\alpha+1} \exp\left(-\frac{\beta}{2\sigma^2}\right), \quad \sigma > 0, \tag{3.2}$$

where $\alpha > 0$ and $\beta > 0$. This density is known as the square-root inverted gamma distribution. When $\beta = 0$ in (3.2), $\pi(\sigma)$ reduces to a general class of improper priors. When $\alpha = 0$ and $\beta = 0$, an improper prior for σ is the Jeffreys (1961) prior. Note that if $\lambda = 1/\sigma^2$, the density function of λ has a gamma distribution with parameters α and $\beta/2$.

By combining (3.1) with (3.2), the posterior density of σ is given by

$$\begin{aligned} \pi(\sigma|\mathbf{Y}) &= \frac{\left[\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta\right]^{n+\alpha}}{2^{n+\alpha-1}\Gamma(n+\alpha)} \left(\frac{1}{\sigma}\right)^{2n+2\alpha+1} \\ &\quad \times \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta\right]\right). \end{aligned} \tag{3.3}$$

It is well known that the Bayes estimator of σ under squared error loss is the posterior mean. It follows from (3.3) that the Bayes estimator of σ is the posterior mean

$$\begin{aligned} \hat{\sigma}_B = E[\sigma|\mathbf{Y}] &= \sqrt{\frac{\left[\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta\right]^{n+\alpha}}{2}} \\ &\quad \times \frac{\Gamma\left(n+\alpha-\frac{1}{2}\right)}{\Gamma(n+\alpha)}. \end{aligned} \tag{3.4}$$

Another problem of interest is to estimating the reliability function $R(t)$ with fixed $t > 0$. Let $R = \exp\left(-t^2/2\sigma^2\right)$ and substituting $\sigma = \left(t^2(-2\log R)^{-1}\right)^{1/2}$ into (3.3), the posterior probability density function of $R = R(t)$ is given by

$$\begin{aligned} \pi(R|\mathbf{Y}) &= \frac{1}{\Gamma(n+\alpha)}\left[\frac{\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta}{t^2}\right]^{n+\alpha} (-\log R)^{n+\alpha-1} \\ &\quad \times R^{\frac{\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta}{t^2} - 1}. \end{aligned} \tag{3.5}$$

From (3.5), the Bayes estimator of $R(t)$ is given by

$$\hat{R}_B = E[R(t)|\mathbf{Y}] = \left[\frac{\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta}{t^2 + \sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta}\right]^{n+\alpha}.$$

The highest posterior density (HPD) estimation is another popular method used by the Bayesian perspective. This parameter estimation is based on the maximum likelihood principle; hence it leads to the mode of the posterior density or HPD estimator. Since the posterior density (3.3) is unimodal, the HPD estimator of σ, denoted by $\hat{\sigma}_H$, is obtained by

$$\hat{\sigma}_H = \sqrt{\frac{\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta}{2(n+\alpha)+1}}.$$

From (3.5), the HPD estimator of $R(t)$, denoted by $\hat{R}_H$ is

$$\hat{R}_H = exp\left(-\frac{(n+\alpha-1)t^2}{\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta - t^2}\right).$$

Remark If we take $m_i = R_i$, $i = 1, \cdots, n-1$ and $k = R_n+1$ then $X(r, n, m, k)$ coincides with the rth progressive type II order statistics from the sample of size n. In this case, the Bayes estimators of the scale parameter σ and the reliability function $R(t)$ with fixed $t > 0$ are given by respectively,

$$\tilde{\sigma}_B = \sqrt{\frac{1}{2}\left[\beta + \sum_{i=1}^{n}(R_i+1)y_i^2\right]}\frac{\Gamma\left(n+\alpha-\frac{1}{2}\right)}{\Gamma(n+\alpha)},$$

and

$$\tilde{R}_B = \left[\frac{\sum_{i=1}^{n}(R_i+1)y_i^2 + \beta}{t^2 + \sum_{i=1}^{n}(R_i+1)y_i^2 + \beta}\right]^{n+\alpha}.$$

3.2 HPD credible intervals

A set $C \subset (0, \infty)$ of the form $C = \{\sigma : \pi(\sigma|\mathbf{y}) \geq c_\alpha\}$, where c_α is the largest constant such that $P(\sigma \in C|\mathbf{y}) = 1-\alpha$, is called a $100(1-\alpha)\%$ HPD credible set for σ. Hence, a $100(1-\alpha)\%$ HPD credible interval choose (c_l, c_u) consist of all values of σ with $\pi(\sigma|\mathbf{y}) > c_\alpha$. Due to the unimodality of (3.3), the $100(1-\alpha)\%$ HPD credible interval (c_l, c_u) for σ must satisfy $\int_{c_l}^{c_u}\pi(\sigma|\mathbf{y})d\sigma = 1-\alpha$ and $\pi(c_u|\mathbf{y}) = \pi(c_l|\mathbf{y})$.

With some algebra, the $100(1-\alpha)\%$ HPD credible interval (c_l, c_u) for σ is given by the simultaneous solution of the equations

$$\gamma(v_1, n+\alpha) - \gamma(v_2, n+\alpha) = 1-\alpha,$$

and

$$\left(\frac{c_u}{c_l}\right)^{2(n+\alpha)+1} = exp(v_1 - v_2),$$

where $v_1 = [\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta]/(2c_l^2)$, $v_2 = [\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta]/(2c_u^2)$, and $\gamma(\cdot,\cdot)$ denotes the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x \frac{1}{\Gamma(a)} z^{a-1} e^{-z} dz.$$

Similarly, the $100(1-\alpha)\%$ HPD credible interval (d_l, d_u) for $R(t)$ must satisfy the following two equations:

$$\gamma(-\delta log d_l, n+\alpha) - \gamma(-\delta log d_u, n+\alpha) = 1-\alpha,$$

and

$$\left(\frac{\log d_u}{\log d_l}\right)^{n+\alpha-1} = \left(\frac{d_l}{d_u}\right)^{\delta-1},$$

where $\delta = [\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta]/t^2$.

3.3 Prediction of future observations

Let $\mathbf{Y} = (Y(1,n,\tilde{m},k), \cdots, Y(n,n,\tilde{m},k))$ be the first n of the generalized order statistics of $\{Y_n, n \geq 1\}$ from the Rayleigh distribution given by (1.2) where $\tilde{m} = (m_1, \cdots, m_{n-1})$. Two sample schemes are used in which informative sample is the first n of generalized order statistics and $X_{(1)} < \cdots < X_{(n)}$ are the order statistics of a future sample. Let X_s, $s = 1, \cdots, n$ be the order statistics in a sample of size n with same distribution.

The p.d.f of the sth $(1 \leq s \leq n)$ order statistics is

$$f(x_s|\sigma) = \frac{N!}{(s-1)!(n-s)!}\frac{x_s}{\sigma^2} exp\left(-\frac{(n-s+1)x_s^2}{2\sigma^2}\right)\left[1 - exp\left(-\frac{x_s^2}{2\sigma^2}\right)\right]^{s-1} \tag{3.6}$$

for $x_s > 0$.

From (3.6) and (3.3), the predictive distribution of X_s given $\mathbf{Y}$ is

$$\begin{aligned} f(x_s|\mathbf{Y}) &= \int_0^\infty f(x_s|\sigma)\pi(\sigma|\mathbf{Y})d\sigma, \\ &= \frac{2n!(n+\alpha)}{(s-1)!(n-s)!}\left(\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta\right)^{n+\alpha} \sum_{l=1}^{s-1}(-1)^l \binom{s-1}{l} \\ &\quad \times x_s\left((n-s+l+1)x_s^2 + \sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta\right)^{-(n+\alpha+1)} \end{aligned} \tag{3.7}$$

for $x_s > 0$.

Under squared error loss, the Bayes predictive estimator of X_s is the expectation of the predictive distribution. Therefore, from (3.8) it follows

$$\hat{X}_s = E(X_s|\mathbf{Y}] = \frac{n!\sqrt{\frac{\pi}{2}}}{(s-1)!(n-s)!}\sum_{l=1}^{s-1}(-1)^l \left(\begin{array}{c} s-1 \\ l \end{array} \right) (n-s+l+1)^{-\frac{3}{2}}\hat{\sigma}_B$$

where $\hat{\sigma}_B$ is the Bayes estimator of θ given by (3.4).

According to (3.6), the predictive reliability function of x_s is given by

$$\begin{aligned} R_s(t|\mathbf{Y}) \quad = \quad & \frac{2n!(n+\alpha)\left(\sum_{i=1}^{n-1}(m_i+1)y_i^2+ky_n^2+\beta\right)^{n+\alpha}}{(s-1)!(n-s)!} \\ & \times \sum_{l=0}^{s-1}(-1)^l \left(\begin{array}{c} s-1 \\ l \end{array} \right) \frac{1}{(n-s+l+1)w_{l,t}^{n+\alpha}} \end{aligned}$$

where $w_{l,t} = \left(\sum_{i=1}^{n-1}(m_i+1)y_i^2 + ky_n^2 + \beta + (n-s+l+1)t\right)^{n+\alpha}$.

The $100(1-\alpha)\%$ HPD credible interval (c_l, c_u) for X_s should simultaneously satisfy $\int_{c_l}^{c_u} f(x_s|\mathbf{y})dx_s = 1-\alpha$ and $f(c_l|\mathbf{y}) = f(c_u|\mathbf{y})$. Therefore, after some algebra, the $100(1-\alpha)\%$ HPD credible interval (c_l, c_u) for X_s is given by the simultaneous solution of the equations

$$\begin{aligned} \frac{n!}{s!(n-s)!}\sum_{j=0}^{s-1}\frac{(-1)^j\left(\begin{array}{c} s-1 \\ l \end{array} \right)}{n-s+j+1}&\left\{\left(1+\frac{(n-s+j+1)c_l^2}{V}\right)^{-(n+\alpha)}\right. \\ &\left. -\left(1+\frac{(n-s+j+1)c_u^2}{V}\right)^{-(n+\alpha)}\right\} = 1-\alpha \end{aligned}$$

and

$$c_l\Psi\left(V, n+\alpha+1; c_l\right) = c_u\Psi\left(V, n+\alpha+1; c_u\right)$$

where $V = \sum_{i=1}^{n-1}(m_i+1)y_i^2+ky_n^2+\beta$ and $\Psi(a,b,c) = \sum_{j=0}^{s-1}(-1)^j \left(\begin{array}{c} s-1 \\ l \end{array} \right) (1+(n-s+j+1)c^2/a)^{-b}$.

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