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Exact Solutions for the Two-Dimensional Incompressible Magnetohydrodynamics Equations

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Abstract

In this paper, we construct three kinds of exact solutions for two-dimensional incompressible magnetohydrodynamics (MHD) equations by making use of the Riccati auxiliary equation. These solutions include well-known bounded soliton solutions.

Mathematcs Subject Classification: 35Q31, 35C05, 76B03, 76M60

Keywords: MHD equations, Riccati auxiliary equation, exact solutions, soliton solutions

1 Introduction

In the present paper, we shall consider the problem of finding exact solutions of the magnetohydrodynamic (MHD) equations

$$\begin{aligned} &\mathrm{div}(\mathbf{u}) = \mathrm{div}(\mathbf{v}) = 0, \\ &\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\left(p + \frac{1}{2}|\mathbf{v}|^2\right) - \nu_1\partial_x^2\mathbf{u} - \nu_2\partial_y^2\mathbf{u} = (\mathbf{v}\cdot\nabla)\mathbf{v}, \\ &\mathbf{v}_t + (\mathbf{u}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{u} - \eta_1\partial_x^2\mathbf{v} - \eta_2\partial_y^2\mathbf{v} = 0, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1(x,y,t), u_2(x,y,t))^T, \mathbf{v} = (v_1(x,y,t), v_2(x,y,t))^T$ and $p = p(x,y,t)$ represent the unknown velocity field, the magnetic field and the pressure, respectively, and $\nu_1, \nu_2, \eta_1, \eta_2$ are parameters [1]-[3]. The dynamics of fluids in electromagnetic fields such as plasma, liquid metals and salt water can be described by MHD equations. Such flows may arise both in laboratory and astrophysical situations and can be often assumed to have an axial symmetry. This system is of interest for various reasons. For example, it includes some known equations, say incompressible Navier-Stokes equation for $\mathbf{v} = 0$ [4]-[7], and incompressible Euler equations for $\mathbf{v} = 0$, $\nu_1 = \nu_2 = 0$ [8]-[13]. Therefore, the study of system (1) can help the understanding of the Navier-Stokes equations and Euler equations. A set of exact analytical solutions of the axisymmetric MHD equations for stationary and incompressible flows was given [14]. The global existence of strong solutions of the regularized MHD system was proved [15]. The regularity criteria to the 2D generalized MHD equations with zero magnetic diffusivity considered [16].

In the following, we construct three kind of exact solutions for the MHD equations including well-known soliton solutions. The key idea of our method is to take full advantage of a Riccati equation involving a parameter and use its solutions to replace tanh-function in tanh method. This paper is organized as follows. In section 2, we search for the exact solutions for the MHD equations (1) by using Riccati auxiliary equation.

2 The exact solutions

For convenience to discuss in the context, we write the MHD system (1) into the following scalar form

$$\begin{aligned}
&u_{1,x} + u_{2,y} = v_{1,x} + v_{2,y} = 0,\\
&u_{1,t} + u_1 u_{1,x} + u_2 u_{1,y} + p_x + v_2 v_{2,x} - v_2 v_{1,y} - \nu_1 u_{1,xx} - \nu_2 u_{1,yy} = 0,\\
&u_{2,t} + u_1 u_{2,x} + u_2 u_{2,y} + p_y + v_1 v_{1,y} - v_1 v_{2,x} - \nu_1 u_{2,xx} - \nu_2 u_{2,yy} = 0,\\
&v_{1,t} + u_1 v_{1,x} + u_2 v_{1,y} - v_1 u_{1,x} - v_2 u_{1,y} - \eta_1 v_{1,xx} - \eta_2 v_{1,yy} = 0,\\
&v_{2,t} + u_1 v_{2,x} + u_2 v_{2,y} - v_1 u_{2,x} - v_2 u_{2,y} - \eta_1 v_{2,xx} - \eta_2 v_{2,yy} = 0.
\end{aligned} \tag{2}$$

To look for the travelling wave solution of system (2), we make transformation

$$u_i(x,y,t) = U_i(\xi), \quad v_i(x,y,t) = V_i(\xi),\ p(x,y,t) = P(\xi),\ i = 1,2,$$

where $\xi = x + \alpha y + \beta t$, and change the system of partial differential (2) into

a system of ordinary differential equations

$$\begin{aligned}
&U_1' + \alpha U_2' = V_1' + \alpha V_2' = 0,\\
&\beta U_1' + U_1 U_1' + \alpha U_2 U_1' + P' + V_2 V_2' - \alpha V_2 V_1' - (\nu_1 + \alpha^2 \nu_2) U_1'' = 0,\\
&\beta U_2' + U_1 U_2' + \alpha U_2 U_2' + \alpha P' - V_1 V_2' + \alpha V_1 V_1' - (\nu_1 + \alpha^2 \nu_2) U_2'' = 0,\\
&\beta V_1' + U_1 V_1' + \alpha U_2 V_1' - V_1 U_1' - \alpha V_2 U_1' - (\eta_1 + \alpha^2 \eta_2) V_1'' = 0,\\
&\beta V_2' + U_1 V_2' + \alpha U_2 V_2' - V_1 U_2' - \alpha V_2 U_2' - (\eta_1 + \alpha^2 \eta_2) V_2'' = 0,
\end{aligned} \tag{3}$$

where $' = \frac{d}{d\xi}$. Balancing the highest order linear terms with nonlinear terms in the system (3) suggests the following ansatz

$$\begin{aligned}
&U_1 = a_0 + a_1\varphi, \quad U_1 = b_0 + b_1\varphi, \quad V_1 = c_0 + c_1\varphi,\\
&V_1 = d_0 + d_1\varphi, \quad P = e_0 + e_1\varphi + e_2\varphi^2,
\end{aligned} \tag{4}$$

where $a_i, b_i, c_i, i = 0, 1$, $e_j, j = 0, 1, 2$, α and β are constants to be determined, and the function φ satisfying a Riccati equation

$$\varphi'^2 + \varphi^2, \quad \varepsilon = \pm 1, \quad k \geq 0, \tag{5}$$

which admits three kinds of general solutions

$$\varphi = -k\tanh(k\xi), \quad -k\coth(k\xi), \quad \text{for} \ \ \varepsilon = -1, \tag{6}$$

$$\varphi = -\frac{1}{\xi}, \quad \text{for} \ \ k = 0, \tag{7}$$

$$\varphi = k\tan(k\xi), \quad -k\cot(k\xi), \quad \text{for} \ \ \varepsilon = 1. \tag{8}$$

Substituting (4) into system (3) and using the Riccati equation (5), then setting the coefficients of all powers of φ to zero, we will get a set of algebraic

system with respect to variables $a_i, b_i, c_i, i = 0, 1$, $e_j, j = 0, 1, 2$, α and β

$$
\begin{aligned}
&a_1 + \alpha b_1 = 0, \quad c_1 + \alpha d_1 = 0,\\
&\beta a_1 + a_0 a_1 + \alpha a_1 b_0 - \alpha c_1 d_0 + d_0 d_1 + e_1 = 0,\\
&a_1^2 + \alpha a_1 b_1 - \alpha c_1 d_1 + d_1^2 + 2e_2 - 2\nu_1 a_1 - 2\alpha^2 \nu_2 a_1 = 0,\\
&\beta b_1 + a_0 b_1 + \alpha b_0 b_1 + \alpha c_0 c_1 - c_0 d_1 + \alpha e_1 = 0,\\
&a_1 b_1 + \alpha b_1^2 + \alpha c_1^2 - c_1 d_1 + 2\alpha e_2 - 2\nu_1 b_1 - 2\alpha^2 \nu_2 b_1 = 0,\\
&-a_1 c_0 + \beta c_1 + a_0 c_1 + \alpha b_0 c_1 - \alpha a_1 d_0 = 0,\\
&\alpha b_1 c_1 - \alpha a_1 d_1 - 2\eta_1 c_1 - 2\alpha^2 \eta_2 c_1 = 0,\\
&-b_1 c_0 + a_0 c_1 - \alpha b_1 d_0 + \beta d_1 + \alpha b_0 d_1 = 0,\\
&a_1 c_1 - b_1 c_1 - 2\eta_1 d_1 - 2\alpha^2 \eta_2 d_1 = 0.
\end{aligned}
$$

From the output of symbolic computation software *Mathematica*, we obtain a solution, namely,

$$
\begin{aligned}
&a_1 = b_1, \ c_1 = d_1, \quad d_1 = \pm b_1, \ e_1 = \mp b_1(c_0 + d_0), \quad e_2 = -b_1^2,\\
&\beta = b_0 - a_0 + d_0 - c_0, \quad \nu_1 = -\nu_2, \quad \eta_1 = -\eta_2,
\end{aligned}
\tag{9}
$$

where a_0, b_0, c_0, d_0 and k are arbitrary constants. Since k is a arbitrary parameter, according to (4), (6)-(8) and (9), we obtain three kinds of travelling wave solutions for the new coupled MHD system (1), namely

a soliton solution with $\varepsilon = -1$

$$
\begin{aligned}
&u_1 = a_0 - k b_1 \tanh(k\xi), \quad u_2 = b_0 - k b_1 \tanh(k\xi),\\
&v_1 = c_0 \pm k b_1 \tanh(k\xi), \quad v_2 = d_0 \pm k b_1 \tanh(k\xi),\\
&p = e_0 \mp k b_1 \tanh(k\xi) - b_1^2 k^2 \tanh^2(k\xi),
\end{aligned}
\tag{10}
$$

a periodic solution with $\varepsilon = 1$

$$u_1 = a_0 + kb_1 \tan(k\xi), \quad u_2 = b_0 + kb_1 \tan(k\xi),$$
$$v_1 = c_0 \pm kb_1 \tan(k\xi), \quad v_2 = d_0 \pm kb_1 \tan(k\xi),$$
$$p = e_0 \mp kb_1 \tan(k\xi) - b_1^2 k^2 \tan^2(k\xi),$$

and a rational solution with $k = 0$

$$u_1 = a_0 - \frac{b_1}{\xi}, \quad u_2 = b_0 - \frac{b_1}{\xi}, \quad v_1 = c_0 \pm \frac{b_1}{\xi},$$
$$v_2 = d_0 \pm \frac{b_1}{\xi}, \quad p = e_0 + \mp\frac{b_1}{\xi} - \frac{b_1^2}{\xi^2},$$

where $\xi = x - y + (b_0 - a_0 + d_0 - c_0)t$.

It is obvious that the first kind of solutions are bounded soliton solutions (10) respect to variables x, y and t. These solutions, to the best of our knowledge, should be previously unknown.

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