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Generalized Tangent Numbers and Polynomials Associated with p-Adic Integral on $\mathbb{Z}_p$

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Abstract

In this paper we introduce the generalized tangent numbers $T_{n,\chi}$ and polynomials $T_{n,\chi}(x)$. Some interesting results and relationships are obtained.

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1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(g) = \int_X g(x)d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [4].} \tag{1.1}$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}g(l), \text{ see [4, 5].} \tag{1.2}$$

Euler numbers, Euler polynomials, and tangent numbers, tangent polynomials were studied by many authors(see [1-11]). Bernoulli numbers, Bernoulli polynomials, Euler numbers, Euler polynomials, tangent numbers, and tangent polynomials possess many interesting properties and arise in many areas of mathematics and physics. These numbers are still in the center of the advanced mathematical research. Especially, in number theory and quantum theory, they have many applications.

First, we introduce the tangent numbers and tangent polynomials(see [6]). The tangent numbers T_n are defined by the generating function:

$$F(t) = \frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n\frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}), \text{ cf. [6]} \tag{1.3}$$

where we use the technique method notation by replacing T^n by $T_n (n \geq 0)$ symbolically. We consider the tangent polynomials $T_n(x)$ as follows:

$$F(x,t) = \left(\frac{2}{e^{2t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}. \tag{1.4}$$

Note that $T_n(x) = \sum_{k=0}^{n}\binom{n}{k}T_k x^{n-k}$. In the special case $x = 0$, we define $T_n(0) = T_n$.

The purpose of this paper is to construct the generalized tangent polynomials $T_{n,\chi}(x)$ attached to χ and derive a new l-series which interpolates the generalized tangent polynomials $T_{n,\chi}(x)$.

2 Generalized tangent polynomials

In this section, our goal is to give generating functions of the generalized tangent numbers and polynomials. These numbers will be used to prove the analytic continuation of the l-series. Let χ be Dirichlet's character with conductor

$d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. Then the generalized tangent numbers associated with associated with χ, $T_{n,\chi}$, are defined by the following generating function

$$F_\chi(t) = \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{2at}}{e^{2dt}+1} = \sum_{n=0}^{\infty} T_{n,\chi}\frac{t^n}{n!}. \tag{2.1}$$

We now consider the generalized tangent polynomials associated with χ, $T_{n,\chi}(x)$, are also defined by

$$F_\chi(x,t) = \left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{2at}}{e^{2dt}+1}\right)e^{xt} = \sum_{n=0}^{\infty} T_{n,\chi}(x)\frac{t^n}{n!}. \tag{2.2}$$

When $\chi = \chi^0$, above (2.1) and (2.2) will become the corresponding definitions of the tangent numbers T_n and polynomials $T_n(x)$.

Since

$$\begin{aligned}\sum_{m=0}^{\infty} T_{m,\chi}(x)\frac{t^m}{m!} &= \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{2at}}{e^{2dt}+1}e^{xt}\\ &= \sum_{a=0}^{d-1}\chi(a)(-1)^a\left(\frac{2e^{(\frac{2a+x}{d})dt}}{e^{2dt}+1}\right)\\ &= \sum_{m=0}^{\infty}\left(d^m\sum_{a=0}^{d-1}\chi(a)(-1)^a T_m\left(\frac{2a+x}{d}\right)\right)\frac{t^m}{m!},\end{aligned}$$

we have the following theorem.

Theorem 2.1 *Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1$(mod 2). Then we have*

(1) $T_{m,\chi}(x) = d^m\sum_{a=0}^{d-1}\chi(a)(-1)^a T_m\left(\frac{2a+x}{d}\right)$,

(2) $T_{m,\chi} = d^m\sum_{a=0}^{d-1}\chi(a)(-1)^a T_m\left(\frac{2a}{d}\right)$,

(3) $T_{m,\chi}(x) = \sum_{l=0}^{m}\binom{m}{l}T_{l,\chi}x^{m-l}$.

For $n \in \mathbb{N}$ with $n \equiv 0 (\mathrm{mod}\ 2)$, we have

$$\begin{aligned}&\frac{-2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{2at}}{e^{2dt}+1}e^{2ndt} + \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{2at}}{e^{2dt}+1}\\ &= \sum_{m=0}^{\infty}\left(2\sum_{a=0}^{nd-1}\chi(a)(-1)^a(2a)^m\right)\frac{t^m}{m!}\end{aligned}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 2.2 *Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, n a positive even integer, and $m \in \mathbb{N}$. Then we have*

$$2\sum_{a=0}^{nd-1}\chi(a)(-1)^a(2a)^m = -T_{m,\chi}(2nd) + T_{m,\chi}.$$

Next, we introduce the l-series and two variable l-series.

Definition 2.3 *For $s \in \mathbb{C}$, define two variable l-series as*

$$l(s,x|\chi) = 2\sum_{m=0}^{\infty}\frac{(-1)^m\chi(m)}{(2m+x)^s}.$$

By using (2.2), we easily see that

$$\begin{aligned}
F_\chi(x,t) &= \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{2at}}{e^{2dt}+1}e^{xt} \\
&= 2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{(2a+x)t}\sum_{l=0}^{\infty}(-1)^l e^{2dlt} \\
&= 2\sum_{a=0}^{d-1}\sum_{l=0}^{\infty}\chi(a)(-1)^{a+dl}e^{(2a+x+2dl)t} \\
&= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m e^{(2m+x)t}.
\end{aligned}$$

Then we have

$$\left(\frac{d}{dt}\right)^k F_\chi(x,t)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}\chi(n)(-1)^n(2n+x)^k, \tag{2.3}$$

and

$$\left(\frac{d}{dt}\right)^k\left(\sum_{n=0}^{\infty}T_{n,\chi}(x)\frac{t^n}{n!}\right)\Big|_{t=0} = T_{k,\chi}(x), \text{ for } k \in \mathbb{N}. \tag{2.4}$$

By (2.3), (2.4), we have the following theorem.

Theorem 2.4 *For any positive integer k, we have*

$$T_{k,\chi}(x) = l(-k,x|\chi).$$

Definition 2.5 *For $s \in \mathbb{C}$, define l-series as*

$$l(s \mid \chi) = 2\sum_{m=1}^{\infty} \frac{(-1)^m \chi(m)}{(2m)^s}.$$

By simple calculation, we have the following theorem.

Theorem 2.6 *For any positive integer k, we have*

$$l(-k \mid \chi) = T_{k,\chi}.$$

3 Witt-type formulae on $\mathbb{Z}_p$ in p-adic number field

Our primary aim in this section is to obtain the Witt-type formulae of the generalized tangent numbers $T_{n,\chi}$ and polynomials $T_{n,\chi}(x)$ attached to χ. Let χ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\bmod\ 2)$. Let $g(y) = \chi(y)e^{(2y+x)t}$. By (1.1), we derive

$$\begin{aligned} I_1\left(\chi(y)e^{(2y+x)t}\right) &= \int_X \chi(y)e^{(2y+x)t} d\mu_{-1}(y) \\ &= \left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{2at}}{e^{2dt}+1}\right)e^{xt} \qquad (3.1) \\ &= \sum_{n=0}^{\infty} T_{n,\chi}(x)\frac{t^n}{n!}. \end{aligned}$$

By using Taylor series of $e^{(2y+x)t}$ in the above equation (3.1), we obtain

$$\sum_{n=0}^{\infty}\left(\int_X \chi(y)(2y+x)^n d\mu_{-1}(y)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n,\chi}(x)\frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the generalized Tangent polynomials attached to χ as follows:

Theorem 3.1 *For positive integers n, we have*

$$T_{n,\chi}(x) = \int_X \chi(y)(2y+x)^n d\mu_{-1}(y). \qquad (3.2)$$

Observe that for $x = 0$, the equation (3.2) reduces to (3.3).

Corollary 3.2 *For positive integers n, we have*

$$T_{n,\chi} = \int_X \chi(y)(2y)^n d\mu_{-1}(y). \tag{3.3}$$

By (3.1) and (1.2), we have the following theorem:

Theorem 3.3 *For positive integers n, we have*

$$\int_X \chi(y)(2y+2nd)^m d\mu_{-1}(y) - (-1)^n \int_X \chi(y)(2y)^m d\mu_{-1}(y)$$
$$= 2^{m+1} \sum_{l=0}^{nd-1} (-1)^{n-1-l} \chi(l) l^m.$$

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