Applied Mathematical Sciences, Vol. 7, 2013, no. 98, 4867 - 4873
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2013.37379

On Some Identities and Generating Functions for *k*-Pell-Lucas Sequence

Paula Catarino [1] and Paulo Vasco [2]

Department of Mathematics
School of Science and Technology
University of Trás-os-Montes e Alto Douro (Vila Real – Portugal)
pcatarin@utad.pt, pvasco@utad.pt

Abstract

We obtain the Binet's formula for *k*-Pell-Lucas numbers and as a consequence we obtain some properties for *k*-Pell-Lucas numbers. Also we give the generating function for the *k*-Pell-Lucas sequences and another expression for the general term of the sequence, using the ordinary generating function, is provided.

Mathematics Subject Classification: 11B37, 05A15, 11B83.

Keywords: Pell numbers, Pell-Lucas numbers, *k*-Pell numbers, *k*-Pell-Lucas numbers, Binet's formula, Generating Functions

1. Introduction

The well-known Fibonacci (and Lucas) sequence is one of the sequences of positive integers that have been studied over several years. Many papers are dedicated to Fibonacci sequence, such as the work of Hoggatt, in [17] and Vorobiov, in [13], among others and more recently we have, for example, the

---

[1] Collaborator of CIDMA and CM-UTAD (Portuguese research centers).

[2] Member of CM-UTAD.

works of Caldwell *et al*. in [4], Marques in [7], Shattuck in [11] and Falcőn *et al*., in [15]. The matrix method is used to get some properties for some sequences of numbers. For example in [15] the authors consider some properties for the *k*-Fibonacci numbers obtained from elementary matrix algebra and its identities including generating function and divisibility properties appears in the paper of Bolat *et al*., in [3]. The sequence of Pell numbers is other sequence of numbers that is defined by the recursive sequence given by $\mathrm{P_n} = 2\mathrm{P_{n-1}} + \mathrm{P_{n-2}}$, $\mathrm{n} \geq 2$, with the initial conditions $\mathrm{P_0} = 0$ and $\mathrm{P_1} = 1$. This sequence has been studied and some of its basic properties are known (see, for example, the study of Horadam, in [2]). In [10], we find the matrix method for generating this sequence and comparable matrix generators have been considered by Kalman, in [6], by Bicknell, in [12], for the Fibonacci and Pell sequences. Also in [16], Koshy study the relation with the Pascal's Triangle and the sequences of Fibonacci, Lucas and Pell numbers. Sometimes, in the literature, are considered other sequences namely, Pell-Lucas and Modified Pell sequences and also Dasdemir, in [1], consider new matrices which are based on these sequences as well as that they have the generating matrices The Pell-Lucas sequence is defined by $\mathrm{Q_n} = 2\mathrm{Q_{n-1}} + \mathrm{Q_{n-2}}$, $\mathrm{n} \geq 2$, with the initial conditions $\mathrm{Q_0} = \mathrm{Q_1} = 2$. The Binet's formula is also well known for several of these sequences. Sometimes some basic properties come from this formula. For example, for the sequence of Jacobsthal number, Koken and Bozkurt, in [8], deduce some properties and the Binet's formula, using matrix method. In [9], Yilmaz et al. study some more properties related with *k*-Jacobsthal numbers.

According Catarino, in [14], and also Jhala *et al*., in [5], we consider, in this paper, the *k*-Pell-Lucas sequence and many properties are proved by easy arguments for the *k*-Pell-Lucas numbers.

2. The *k*-Pell-Lucas sequence and some properties

For any positive real number k, the *k*-Pell sequence say $\{P_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by $P_{k,n+1} = 2P_{k,n} + \mathrm{k}P_{k,n-1}$, for $n \geq 1$, with the initial conditions given by $P_{k,0} = 0$, $P_{k,1} = 1$ (see [14]). Now we define the *k*-Pell-Lucas $\{Q_{k,n}\}_{n \in \mathbb{N}}$ sequence satisfying the recursive recurrence given by

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \text{ for } n \geq 1, \quad (1)$$

with the initial conditions given by $Q_{k,0} = 2$, $Q_{k,1} = 2$. (2)

Next we find the explicit formula for the term of order n of the *k*-Pell-Lucas sequence using the well-known results involving recursive sequences. The characteristic equation $r^2 - 2r - k = 0$, associated to the recurrence relation (1), has two distinct roots r_1 and r_2 being $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$, where k is a real positive number. Since $\sqrt{1+k} > 1$, then $r_2 < 0$ and so $r_2 < 0 < r_1$. Also we obtain $r_1 + r_2 = 2$, $r_1 - r_2 = 2\sqrt{1+k}$ and $r_1 r_2 = -k$.

Proposition 1 (Binet's formula)
The nth k-Pell-Lucas number is given by

$$Q_{k,n} = r_1^n + r_2^n \tag{3}$$

where r_1 *and* r_2 *are the roots of the characteristic equation and* $r_1 > r_2$.

Proof: Since the characteristic equation has two distinct roots, the sequence

$$Q_{k,n} = c_1(r_1)^n + c_2(r_2)^n \tag{4}$$

is the solution of the equation (1). Giving to n the values $n = 0$ and $n = 1$ and solving this system of linear equations, we obtain a unique value for c_1 and c_2. So, we get the following distinct values, $c_1 = 1$ and $c_2 = 1$. Now, using (4), we obtain (3) as required.

■

Proposition 2 (Catalan's identity)

$$Q_{k,n-r}Q_{k,n+r} - Q_{k,n}^2 = (-k)^{n-r}\left(Q_{k,r}^2 - 4(-k)^r\right) \tag{5}$$

Proof: Using the Binet's formula (3) and the fact that $r_1 r_2 = -k$, we get

$$\begin{aligned} Q_{k,n-r}Q_{k,n+r} - Q_{k,n}^2 &= (r_1^{n-r} + r_2^{n-r})(r_1^{n+r} + r_2^{n+r}) - (r_1^n + r_2^n)^2 \\ &= (r_1 r_2)^n (r_1^{-r} r_2^r + r_1^r r_2^{-r} - 2) \\ &= (-k)^{n-r}\left((r_1^r + r_2^r)^2 - 4(-k)^r\right) \\ &= (-k)^{n-r}\left(Q_{k,r}^2 - 4(-k)^r\right), \end{aligned}$$

that is, the identity required.

■

Note that for $r = 1$ in Catalan's identity obtained, we get the Cassini's identity for the k-Pell-Lucas numbers sequence. In fact, the equation (5) for $r = 1$, yields

$$Q_{k,n-1}Q_{k,n+1} - Q_{k,n}^2 = (-k)^{n-1}\left(Q_{k,1}^2 - 4(-k)\right)$$

and using one of the initial conditions of this sequence we obtain the following result.

Proposition 3 (Cassini's identity)

$$Q_{k,n-1}Q_{k,n+1} - Q_{k,n}^2 = 4(-k)^{n-1}(1+k) \tag{6}$$

■

The d'Ocagne's identity can also be obtained using the Binet's formula as it was done in [5] for the k-Jacobsthal sequence and in [14] for the k-Pell sequence, although in this case, this identity is not so simple. Hence we have

Proposition 4 (d'Ocagne's identity)
If $m > n$ *then*

$$Q_{k,m}Q_{k,n+1} - Q_{k,m+1}Q_{k,n} = (-1)^n k^n 2\sqrt{1+k}\left(Q_{k,m-n} - 2\left(1+\sqrt{1+k}\right)^{m-n}\right) \tag{7}$$

Proof: Once more, using the Binet's formula (3), the fact that $r_1 r_2 = -k$ and $m > n$, we get

$$\begin{aligned} Q_{k,m}Q_{k,n+1} - Q_{k,m+1}Q_{k,n} &= (r_1^m + r_2^m)(r_1^{n+1} + r_2^{n+1}) - (r_1^{m+1} + r_2^{m+1})(r_1^n + r_2^n) \\ &= (r_1 r_2)^n (r_1 - r_2)(r_2^{m-n} - r_1^{m-n}) \\ &= (r_1 r_2)^n (r_1 - r_2)\left(Q_{k,m-n} - 2r_1^{m-n}\right) \\ &= (-k)^n \left(2\sqrt{1+k}\right)\left(Q_{k,m-n} - 2r_1^{m-n}\right) \\ &= (-k)^n \left(2\sqrt{1+k}\right)\left(Q_{k,m-n} - 2\left(1+\sqrt{1+k}\right)^{m-n}\right) \end{aligned}$$

and the result follows.

■

Again using the Binet's formula (3) we obtain other property of the k-Pell-Lucas sequence which is stated in the following proposition.

Proposition 5

$$\lim_{n\to\infty} \frac{Q_{k,n}}{Q_{k,n-1}} = r_1. \tag{8}$$

Proof: We have that

$$\lim_{n\to\infty} \frac{Q_{k,n}}{Q_{k,n-1}} = \lim_{n\to\infty} \left(\frac{r_1^n + r_2^n}{r_1^{n-1} + r_2^{n-1}}\right) \tag{9}$$

Using the ratio $\frac{r_2}{r_1}$ and since $\left|\frac{r_2}{r_1}\right| < 1$, then $\lim_{n\to\infty} \left(\frac{r_2}{r_1}\right)^n = 0$. Next we use this fact writing (9) with an equivalent form using this ratio, obtaining

$$\lim_{n\to\infty} \frac{Q_{k,n}}{Q_{k,n-1}} = \lim_{n\to\infty} \frac{1+\left(\frac{r_2}{r_1}\right)^n}{\frac{1}{r_1}+\left(\frac{r_2}{r_1}\right)^n \frac{1}{r_2}} = \frac{1}{\frac{1}{r_1}} = r_1.$$

■

Also, we easily can show the following result using basic tools of calculus of limits and (8).

Proposition 6

$$\lim_{n\to\infty} \frac{Q_{k,n-1}}{Q_{k,n}} = \frac{1}{r_1}. \tag{10}$$

■

3. Generating functions for the k-Pell-Lucas sequences

Next we shall give the generating functions for the k-Pell-Lucas sequence. We will show that k-Pell-Lucas sequence can be considered as the coefficients of the power series of the corresponding generating function.

Let us suppose that the k-Pell-Lucas numbers of order k are the coefficients of a power series centered at the origin, and let us consider the corresponding analytic function q_k defined by

$$q_k(x) = Q_{k,0} + Q_{k,1}x + Q_{k,2}x^2 + \cdots + Q_{k,n}x^n + \cdots, \quad (11)$$

and called the generating function of the k-Pell-Lucas numbers.
Using the initial conditions (2), we get

$$q_k(x) = 2 + 2x + \sum_{n=2}^{\infty} Q_{k,n}x^n \quad (12)$$

Now from (1) we can write (12) as follows

$$q_k(x) = 2 + 2x + \sum_{n=2}^{\infty}(2Q_{k,n-1} + kQ_{k,n-2})x^n. \quad (13)$$

Consider the right side of the equation (13) and doing some calculations, we obtain that

$$2 + 2x + \sum_{n=2}^{\infty}(2Q_{k,n-1} + kQ_{k,n-2})x^n$$
$$= 2 + 2x + 2\sum_{n=2}^{\infty} Q_{k,n-1}x^n + k\sum_{n=2}^{\infty} Q_{k,n-2}x^n$$
$$= 2 + 2x + 2x\sum_{n=2}^{\infty} Q_{k,n-1}x^{n-1} + kx^2\sum_{n=2}^{\infty} Q_{k,n-2}x^{n-2} . \quad (14)$$

Consider that $j = n - 2$ and $p = n - 1$. Then (14) can be written by

$$2 + 2x + 2x(\sum_{p=0}^{\infty} Q_{k,p}x^p - Q_{k,0}) + kx^2\sum_{j=0}^{\infty} Q_{k,j}x^j$$
$$= 2 + 2x + 2x\sum_{p=0}^{\infty} Q_{k,p}x^p + kx^2\sum_{j=0}^{\infty} Q_{k,j}x^j - 4x$$
$$= 2 - 2x + 2x\sum_{p=0}^{\infty} Q_{k,p}x^p + kx^2\sum_{j=0}^{\infty} Q_{k,j}x^j.$$

Therefore, $q_k(x) = 2 - 2x + 2xq_k(x) + kx^2q_k(x)$, which is equivalent to $q_k(x)(1 - 2x - kx^2) = 2 - 2x$, and then the ordinary generating function of the k-Pell-Lucas sequence can be written as

$$q_k(x) = \frac{2-2x}{1-2x-kx^2} . \quad (15)$$

Now, note that if for some sequence $(a_n)_n$, it is known that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L$, being L a positive real number, then considering the power series $\sum_{n=0}^{\infty} a_n x^k$, its radius of convergence R is equal to $\frac{1}{L}$. So, for the k-Pell-Lucas sequence, using (8) and then (10) we know that it can be written as a power series with radius of convergence equal to $\frac{1}{r_1}$. Next we give another expression for the general term of the k-Pell-Lucas sequence using the ordinary generating function.

Proposition 7

Let us consider $q(x) = \sum_{n=0}^{\infty} Q_{k,n}x^n$, *for* $x \in \left]-\frac{1}{r_1}, \frac{1}{r_1}\right[$. *Then we have that*

$$Q_{k,n} = \frac{q^{(n)}(0)}{n!}, \tag{16}$$

where $q^{(n)}(x)$ *denotes the nth order derivative of the function* $q(x)$.

Proof: We have that $Q_{k,0}x^0 = 2$ and

$$q'(x) = \sum_{n=1}^{\infty} nQ_{k,n}\, x^{n-1} = 2 + \sum_{n=2}^{\infty} Q_{k,n}\, x^{n-1}.$$

Also,

$$q^{(2)}(x) = \sum_{n=2}^{\infty} n(n-1)Q_{k,n}\, x^{n-2} = 2 \cdot 1 \cdot Q_{k,2} + \sum_{n=3}^{\infty} n(n-1)Q_{k,n}\, x^{n-2};$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$q^{(l)}(x) = \sum_{n=l}^{\infty} n(n-1)\cdots(n-(l-1))Q_{k,n}\, x^{n-l}.$$

So, $q^{(l)}(x) = l(l-1)\cdots\big(l-(l-1)\big)Q_{k,l} + \sum_{n=l+1}^{\infty} n(n-1)\cdots(n-(l-1))Q_{k,n}\, x^{n-l}$ and then we get $q^{(l)}(x) = l!\, Q_{k,l} + \sum_{n=l+1}^{\infty} n(n-1)\cdots(n-(l-1))Q_{k,n}\, x^{n-l}$. Therefore, $q^{(l)}(0) = l!\, Q_{k,l}$ or $Q_{k,l} = \frac{q^{(l)}(0)}{l!}$. Hence, for all $n \geq 1$, we have that $Q_{k,n} = \frac{q^{(n)}(0)}{n!}$, as required.

■

References

[1] A. Dasdemir, "On the Pell, Pell-Lucas and Modified Pell Numbers by Matrix Method", *Applied Mathematical Sciences*, Vol. 5, no. 64 (2011), 3173-3181.

[2] A. F. Horadam, "Pell identities", *The Fibonacci Quarterly*, 9 (3) (1971) 245-252, 263.

[3] C. Bolat, H. Kőse, "On the Properties of *k*-Fibonacci Numbers", *Int. J. Contemp. Math. Sciences*, 22(5) (2010), 1097-1105.

[4] C. K. Caldwell, T. Komatsu, "Some Periodicities in the Continued Fraction Expansions of Fibonacci and Lucas Dirichlet Series". *Fibonacci Quart*. 48, no. 1 (2010), 47-55.

[5] D. Jhala, K. Sisodiya, G. P. S. Rathore, "On Some Identities for k-Jacobsthal Numbers", *Int. Journal of Math. Analysis*, Vol. 7, no. 12 (2013), 551-556.

[6] D. Kalman, "Generalized Fibonacci numbers by matrix methods", *The Fibonacci Quarterly*, 20(1) (1982), 73-76.

[7] D. Marques, "The Order of Appearance of the Product of Consecutive Lucas Numbers", *The Fibonacci Quarterly*, 51(1) (2013), 38-43.

[8] F. Koken, D. Bozkurt, "On the Jacobsthal numbers by matrix methods", *Int. J. Contemp.Math. Sciences*, 30 (3) (2008), 605-614.

[9] F. Yilmaz, D. Bozkurt, "The Generalized Order-*k* Jacobsthal Numbers", *Int. J. Contemp. Math. Sciences*, 34 (4) (2009), 1685-1694.

[10] J. Ercolano "Matrix generators of Pell sequences", *The Fibonacci Quarterly*, 17 (1) (1979), 71-77.

[11] M. Shattuck, "Combinatorial Proofs of Determinant Formulas for the Fibonacci and Lucas Polynomials", *The Fibonacci Quarterly,* 51(1) (2013), 63-71.

[12] N. Bicknell, "A primer on the Pell sequence and related sequence", *The Fibonacci Quarterly,* 13 (4) (1975), 345-349.

[13] N. N. Vorobiov, *Números de Fibonacci*, Editora MIR, URSS, 1974.

[14] P. Catarino, "On some identities and generating functions for *k*-Pell numbers", *Int. Journal of Math. Analysis,* Vol. 7 no. 38 (2013), 1877-1884.

[15] S. Falcőn, À. Plaza, "On the Fibonacci *k*-numbers", *Chaos, Solitons & Fractals*, 32 (5) (2007), 1615-1624.

[16] T. Koshy, "Fibonacci, Lucas, and Pell numbers, and Pascal's triangle", *Mathematical Spectrum;2010/2011,* Vol. 43 Issue 3 (2011), 125-132.

[17] V. E. Hoggatt, *Fibonacci and Lucas Numbers*. A publication of the Fibonacci Association. University of Santa Clara, Santa Clara. Houghton Mifflin Company, 1969.

Received: July 9, 2013