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A Note on the Weakly Mixing of Semigroup Actions[1]

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Abstract

This paper deals with the weak mixing of semigroup actions. We show that a system of semigroup action (S, X) is weakly mixing iff it has a uniform positive sequence entropy and it is topologically transitive, where X is a compact Hausdorff space and S is an abelian monoid semigroup.

Keywords: semigroup action; weakly mixing; uniform positive sequence entropy

1 Introduction

A *topological dynamical system* in the present article is a triple (S, X, π), where S is a topological semigroup, X is at least a compact Hausdorff space and

$$\pi : S \times X \to X, (s, x) \mapsto sx$$

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is a surjective continuous action with the property that $t(sx) = (ts)x$ for all $x \in X$, $t, s \in S$. Sometimes we call it as a S-system, write it as (S, X). If $S = \{T^n : n = 0, 1, 2, \ldots\}$ and $T : X \to X$ is a continuous surjective map, then the classical dynamical system (S, X) is called a *cascade*. We use the standard notation: (X, T). Moreover to avoid uninteresting cases we assume that S and X are infinite.

Let (S, X) be a S-system. We call (S, X) is *topologically transitive*, if for any given nonempty open subsets U, V of X, there exists $s \in S$, such that $U \cap s^{-1}V \neq \emptyset$; We call (S, X) is *weakly mixing*, if for any given nonempty open subsets U_1, V_1, U_2, V_2 of X, there is $s \in S$, such that $(U_1 \times U_2) \cap (s \times s)^{-1}(V_1 \times V_2) \neq \emptyset$.

We need the following definition.

Let (S, X) be a S-system. An infinite sequence $A = \{s_1, s_2, \ldots, s_i, \ldots\} \subset S$, $\mathcal{U}$ is a finite open cover of X. Let

$$h_A(S, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{n} log(N(\vee_{i=1}^{n} s_i^{-1}\mathcal{U})),$$

where $N(\vee_{i=1}^{n} s_i^{-1}\mathcal{U}) = \min\{|\alpha| : \alpha$ is the finite subcover $\vee_{i=1}^{n} s_i^{-1}\mathcal{U}\}$, $|\alpha|$ is the cardinality of α.

The following definition is defined in [1][2].

Definition 1.1 *Let (S, X) be a S-system.*

(1)We call $(x_1, x_2) \in X \times X$ is a sequence entropy pair of (S, X), if $x_1 \neq x_2$, and for any disjoint closed neighborhoods U_i of points x_i, respectively, where $i = 1,\ 2$, there exists an infinite sequence $A = \{s_1, s_2, \ldots, s_i, \ldots\} \subset S$, such that $h_A(S, \mathcal{U}) > 0$, where $\mathcal{U} = \{U_1^c, U_2^c\}$. Let $SE(S, X)$ denote the set of sequence entropy pairs of (S, X);

(2)We call (S, X) has a uniform positive sequence entropy, if $X \times X \backslash \triangle_X = SE(S, X)$, where $\triangle_X = \{(x, x) | x \in X\}$;

(3)We call $(x_1, x_2) \in X \times X \backslash \triangle_X$ is a weakly mixing pair, if $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$ for any open neighborhood U_i of x_i, where $i = 1, 2$. Let $WM(S, X)$ denote the set of weakly mixing pairs of (S, X).

In the classical topologically dynamical system, W. Huang and X. Ye in [2] showed the following result:

Theorem 1.1 [2] *Let X be a compact metric space, $T : X \to X$ be a continous surjective map. Then the following statements are equivalent:*

(1) (X, T) is weakly mixing;

(2) $WM(X,T) = X \times X \backslash \triangle_X$*;*
(3) (X,T) *has a uniform positive sequence entropy.*

Recently, the dynamical system of semigroup actions was interested by us, for example, X. Yan and L. He[3] have studied the sensitivity of semigroup actions. In this article we derive a similar result to Theorem 1.1, for the actions of more general semigroups. Our main result is Theorem 1.2.

Theorem 1.2 *Let* (S,X) *be a* S*-system, where* X *is a no isolated point and compact* $Hausdorff$ *space.* S *is an abelian monoid semigroup. Then the following statements are equivalent:*

(1) (S,X) *is weakly mixing;*
(2) $WM(S,X) = X \times X \backslash \triangle_X$*;*
(3) (S,X) *has a uniform positive sequence entropy and it is topologically transitive.*

Remark The Theorem 8.6 in [1] pointed out that (S,X) is weakly mixing iff (S,X) has a uniform positive sequence entropy, but we do not see the detailed proof in [1]. The proof of Theorem 1.2 in this article is simple.

2 Preliminaries

Now, we recall some notions of semigroup. A semigroup S is called a *abelian semigroup*, if $s_1 s_2 = s_2 s_1$ for any $s_1, s_2 \in S$; A semigroup S is called *monoid semigroup*, if S has an identity $e \in S$.

Let (S,X) be a S -system, U, V be nonempty open subsets of X, $s \in S$. Denote

$$s^{-1}U = \{x \in X : sx \in U\};$$

$$N(U,V) = \{s \in S : U \cap s^{-1}V \neq \emptyset\}.$$

Definition 2.1[4] [5] *We call a set* $P \subset S$ *is thick, if for any finite set* $F \subset S$*, there exists* $s \in S$ *such that* $Fs \subset P$*.*

Definition 2.2[5] *Let* (S,X) *be a* S*-system.*

(1) (S,X) *is called thick transitive, if* $N(U,V)$ *is thick for any nonempty open subsets* U, V *of* X*;*

(2) (S, X) *is called thick center, if* $N(U, U)$ *is thick for any nonempty open subset* U *of* X.

Definition 2.3 [6] *Let* S *be a semigroup.*

(a) A collection of nonempty sets of S *is called a filter, if it satisfies:*
(1) $\emptyset \notin P$;
(2) if $F_1 \in P$ *and* $F_1 \subseteq F_2$, *then* $F_2 \in P$;
(3) for any $F_1, F_2 \in P$, $F_1 \bigcap F_2 \in P$.
(b) A collection of nonempty sets of S *is called a filter base, if* $[\beta] = \{A :$ *there exists a* $B \in \beta$ *such that* $A \supset B\}$ *is filter.*

The following Theorem is showed in [5], where the equivalence of (1) and (4) is proved in [6].

Theorem 2.4[5] *Let* (S, X) *be a* S*-system, where* S *is an abelian monoid semigroup. Then the following statements are equivalent:*

(1) (S, X) *is weakly mixing;*
(2) $N(U, U) \cap N(U, V) \neq \emptyset$ *for any nonempty open sets* U, V *of* X;
(3)$N(U, U) \cap N(V, U) \neq \emptyset$ *for any nonempty open sets* U, V *of* X;
(4)$\{N(U, V)$: U, V *are nonempty open sets of* X $\}$ *is a filter base;*
(5) (S, X) *is thick transitive;*
(6) (S, X) *is transitive and thick center.*

3 The proof of Theorem 1.2

Definition 3.1[1] *Let* (S, X) *be a* S*-system. For a tuple* $\mathcal{A} = (U_1, U_2, \ldots, U_k)$ *of subsets of* X, *where* $U_i \subset X$ *for* $i = 1, 2, \ldots, n$. $F \subset S$ *is called an independent set for* $\mathcal{A}$, *if for every nonempty finite subset* $J = \{s_1, s_2, \ldots, s_n\} \subset F$, *we have*

$$\bigcap_{i=1}^{n} s_i^{-1} U_{t(i)} \neq \emptyset$$

for all $t = (t(1), t(2), \ldots, t(n)) \in \{1, 2, \ldots, k\}^n$.

Definition 3.2 [1] *Let* (S, X) *be a* S*-system. We call a pair* $x = (x_1, x_2) \in X^2 \backslash \triangle_X$ *is an* IN *pair, if for any product neighborhood* $U_1 \times U_2$ *of* x, *the tuple* (U_1, U_2) *has arbitrarily large finite independence sets.*

The following Theorem is the Theorem 5.9 in [1].

Theorem 3.3 [1] *Let (S, X) be a S-system. Let $(x_1, x_2) \in X^2 \backslash \triangle_X$. Then (x_1, x_2) is a sequence entropy pair if and only if it is an IN pair.*

The proof of Theroem1.2 (1) $\Rightarrow$ (2). Assume that x_1, x_2 are any two different points of X. Let U, V be any open neighborhood of x_1, x_2, respectively. Because (S, X) is weakly mixing, then $N(U, U) \cap N(U, V) \neq \emptyset$ by Theorem 2.4(2). Thus (x_1, x_2) is weakly mixing pair of (S, X).

(2) $\Rightarrow$ (1). For any nonempty subsets U, V of X, we choose $x_1 \in U, x_2 \in V$, and $x_1 \neq x_2$. By the definition of weakly mixing pair, we get

$$N(U, U) \cap N(U, V) \neq \emptyset.$$

So (S, X) is weakly mixing.

(3) $\Rightarrow$ (2). Assume that (x_1, x_2) is a sequence entropy pair of (S, X), now we will show (x_1, x_2) is a weakly mixing pair. Since (x_1, x_2) is a sequence entropy pair, thus (x_1, x_2) is a IN pair by Theorem 3.3. Let U_1, U_2 be any open neighborhood of x_1, x_2, respectively. Thus there is a independent set $J = \{s_1, s_2\}$ for (U_1, U_2), such that $\bigcap_{j=1}^{2} s_j^{-1} U_{t(j)} \neq \emptyset$ for any $t = (t(1), t(2)) \in \{1, 2\}^2$. So we get

$$s_1^{-1}U_1 \cap s_2^{-1}U_1 \neq \emptyset,$$
$$s_1^{-1}U_1 \cap s_2^{-1}U_2 \neq \emptyset.$$

Since (S, X) is also transitive, thus there exists a $s \in S$ such that

$$s_1^{-1}U_1 \cap s_2^{-1}U_1 \cap s^{-1}(s_1^{-1}U_1 \cap s_2^{-1}U_2) \neq \emptyset.$$

So we get

$$\begin{aligned}
& s_1^{-1}(U_1 \cap s^{-1}U_1) \cap s_2^{-1}(U_1 \cap s^{-1}U_2) \\
= \; & (s_1^{-1}U_1 \cap s^{-1}s_1^{-1}U_1) \cap (s_2^{-1}U_1 \cap s^{-1}s_2^{-1}U_2) \\
= \; & s_1^{-1}U_1 \cap s_2^{-1}U_1 \cap s^{-1}s_1^{-1}U_1 \cap s^{-1}s_2^{-1}U_2 \\
= \; & s_1^{-1}U_1 \cap s_2^{-1}U_1 \cap s^{-1}(s_1^{-1}U_1 \cap s_2^{-1}U_2). \\
\neq \; & \emptyset.
\end{aligned}$$

Thus, $U_1 \cap s^{-1}U_1 \neq \emptyset$, $U_1 \cap s^{-1}U_2 \neq \emptyset$. This implies that $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$, then (x_1, x_2) is a weakly mixing pair of (S, X).

(1) $\Rightarrow$ (3). By (S,X) is weakly mixing, we will prove (S,X) has a uniform sequence positive entropy. For any two different points $x_1, x_2 \in X$, we only need to prove that (x_1, x_2) is an IN pair of (S,X) by Theorem 3.3.

Let U_1, U_2 be any open neighborhood of x_1, x_2, respectively, by Theorem 2.4(4)(5), so there exists

$$s_1 \in N(U_1,U_1) \cap N(U_1,U_2) \cap N(U_2,U_1) \cap N(U_2,U_2).$$

It means

$$U_{t(0)} \cap s_1^{-1} U_{t(1)} \neq \emptyset,$$

for any $t = (t(0), t(1)) \in \{1,2\}^2$.

Because (S,X) is weakly mixing, then

$$N(U_{t(0)} \cap s_1^{-1} U_{t(1)}, U_1) \bigcap N(U_{t(0)} \cap s_1^{-1} U_{t(1)}, U_2) \neq \emptyset,$$

for any $t = (t(0), t(1)) \in \{1,2\}^2$.

By the Theorem 2.4(4)(5), there exists $s_2 \in S$ and $s_2 \neq s_1$, such that

$$U_{t(0)} \cap s_1^{-1} U_{t(1)} \cap s_2^{-1} U_1 \neq \emptyset,$$

$$U_{t(0)} \cap s_1^{-1} U_{t(1)} \cap s_2^{-1} U_2 \neq \emptyset.$$

for any $t = (t(0), t(1)) \in \{1,2\}^2$.

That is,

$$U_{t(0)} \cap s_1^{-1} U_{t(1)} \cap s_2^{-1} U_{t(2)} \neq \emptyset,$$

for any $t = (t(0), t(1), t(2)) \in \{1,2\}^3$.

Now we may assume that there is a finite set $J = \{s_1, s_2, \ldots, s_l\}$ such that

$$U_{t(0)} \cap s_1^{-1} U_{t(1)} \cap \cdots \cap s_l^{-1} U_{t(l)} \neq \emptyset,$$

for any $t = (t(0), t(1), t(2), \ldots, t(l)) \in \{1,2\}^{l+1}$.

Because (S,X) is weakly mixing, so we have

$$N(U_{t(0)} \cap s_1^{-1} U_{t(1)} \cap \cdots \cap s_l^{-1} U_{t(l)}, U_1) \bigcap N(U_{t(0)} \cap s_1^{-1} U_{t(1)} \cap \cdots \cap s_l^{-1} U_{t(l)}, U_2) \neq \emptyset,$$

for any $t = (t(0), t(1), t(2), \ldots, t(l)) \in \{1,2\}^{l+1}$. By Theorem 2.4(4)(5), there exists $s_{l+1} \in S$ and $s_{l+1} \notin J$, such that

$$U_{t(0)} \cap s_1^{-1} U_{t(1)} \cap \cdots \cap s_l^{-1} U_{t(l)} \cap s_{l+1}^{-1} U_{t(l+1)} \neq \emptyset.$$

for any $t = (t(0), t(1), t(2), \ldots, t(l), t(l+1)) \in \{1,2\}^{l+2}$.

So, $\{s_1, s_2, \ldots, s_{l+1}\}$ is independence sets of length $l+1$ for (U_1, U_2).

By the induction, we know that (x_1, x_2) is an IN pair of (S,X).

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