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On the FE-Analysis of Inverse Problems: An Application in Electrical Engineering

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Abstract

Studying engineering problems by mathematical modelling and numerical analysis often leads to so-called ill-posed problems. Here we consider Cauchy-type problems based on the application of hybrid insulation. We formulate three optimal control problems, with Neumann or Dirichlet control, on the unit square and use Nitsche's method for handling Dirichlet boundary conditions. For the regularisation we use Tikhonov-techniques.

Mathematics Subject Classification: 65N30

Keywords: finite element method, inverse problem, Cauchy problem

1 Introduction

Studying engineering problems by mathematical modelling and numerical analysis often leads to so-called ill-posed problems. In this note, the case of partial differential equations with conflicting boundary conditions is considered.

Following Louis [5] a prototypical example in classical notation reads

$$\begin{aligned} -\Delta u &= 0\,, && \text{on } \Omega = (0,1)^2\,, \\ \partial_n u &= 0\,, && \text{on } \Gamma_N = \{\, x \in \Omega \mid x_2 = 0 \text{ or } x_2 = 1 \,\}\,, \\ \partial_n u &= f\,, \\ u &= 0\,, && \text{on } \Gamma_O = \{x \in \Omega \mid x_1 = 1\}\,. \end{aligned} \tag{1.1}$$

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In order to show that the underlying solution operator cannot be continuous, which is due to the incompatible prescriptions of $\partial_n u$ and u on Γ_O, one chooses a sequence of boundary data f_k with $\|f_k\| \to 0$ for $k \to \infty$, such that for the corresponding solutions u_k there holds $|u_k| \to \infty$ for $k \to \infty$. One possible choice for data and corresponding solution is

$$f_k = k^{-1} \cos(k\pi y)\,, \quad u_k = k^{-2} \cos(k\pi y) \sinh(k\pi(1-x))\,. \tag{1.2}$$

A picture of data and solution (scaled for graphical output by dividing by 80.000) for the case $k = 10$ can be seen in Figure 1 indicating the instable development of u_k for small data f_k.

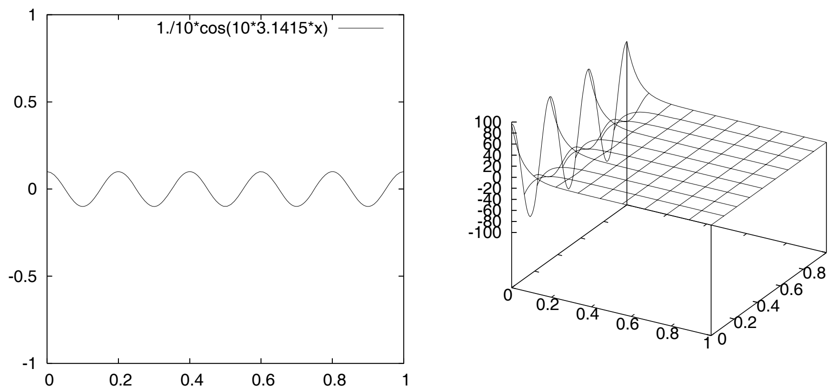

Figure 1: Picture of data and solution (scaled for graphical output by dividing by 80.000) for the case $k = 10$, indicating the instable development of u_k for small data f_k.

The underlying model problem of the present paper is the system (1.1) with additional prescriptions on $\Gamma_C = \{x \in \Omega \mid x_1 = 0\}$.

Depending on the prescriptions on Γ_C relaxed formulations are obtained by transferring the original problem to optimal control problems, e.g.:

Presciption: $\partial_n u = q$ *on* Γ_C

$$J(u,q) \to \min!\,, \qquad J(w,\tau) := \frac{1}{2}\|\partial_n w - f\|^2_{\Gamma_O} \tag{1.3}$$

under the PDE-constraint

$$\begin{aligned} -\Delta u &= 0 && \text{on} \quad \Omega\,, \\ \partial_n u &= 0 && \text{on} \quad \Gamma_N\,, \\ u &= 0 && \text{on} \quad \Gamma_O\,, \\ \partial_n u &= q && \text{on} \quad \Gamma_C\,. \end{aligned} \tag{1.4}$$

Above $J(w,\tau)$ defines the so-called cost functional and q denotes a control variable.

Below we will list further, alternative relaxed settings. Subsequently, the focus is more on the discussion of suitable variational formulations and their corresponding discretisation.
These techniques are incorporated into standard frameworks for inverse problems, as e.g. regularisation approaches by Tikhonov.

1.1 Physical Background, Application

In collaboration with scientists from electrical engineering, we provide the numerical analysis for problems arising in the field of hybrid insulation.

The measurements are performed by F. Mauseth [6] at the university of Trondheim (NTNU). The experimental situation is as follows: we have two electrodes covered with a thick (several milimeter) dieelectric coating between which the breakdown voltage of an air gap can be improved considerably. If free charges are available in the air volume surrounding the structure or in the gap, charges accumulate in the dieelectric surfaces due to eletrostatic attraction.

The charge formation on the insulation surface builds up an electric field that reduces the field in the air gap and increases the field in the solid insulation. The net result is an overall increased insulation performance. In the future this technique may be used to design a construction of compact high voltage equipment.

The surface charge densities associated with this kind of insulation system is quite high, this means a large distance between surface and measuring probe is necessary. Caused by this a direct reading of the surface charge density is not possible since surrounding areas will influence the read out. To circumvent these difficulties, as depicted in Figure 2 (left), the electrical field along a cylinder around the rod was measured.

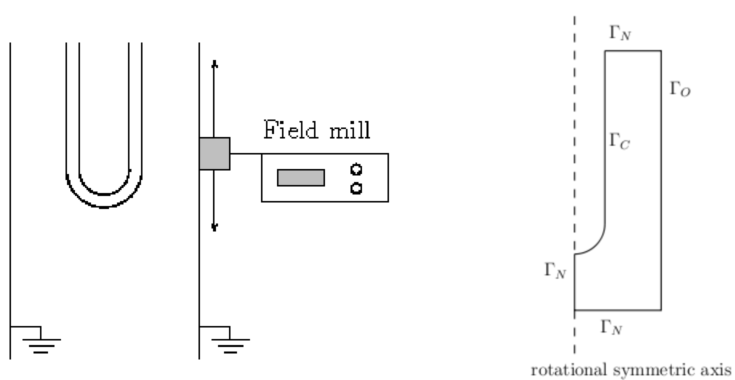

Figure 2: Cylinder with field mill (left) and the two dimensional geometry for the calculation (right).

Exploiting the underlying symmetry by introducing cylinder coordinates we have to solve the two dimensional Laplace equation:

$$-\frac{1}{r}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.5}$$

with the boundary conditions $\partial_n u = 0$ on Γ_N and $u = 0$ on Γ_O on the geometry shown in Figure 2 (right).
Now, with additional information of u along Γ_O provided by measurements given in the form of $\partial_n u = f$ knowledge of u along Γ_C has to be gained. To be more precisely, in other terms we have to determine $\partial_n u$ or alternatively u along Γ_C.

2 OPTIMAL CONTROL PROBLEMS

In this section, motivated by our application, we consider the task of finding (stable) solutions of the system (1.1) and extract information about their distributions q along Γ_C in possible forms $q = \partial_n u$ or $q = u$.
To this end, we first list different relaxed settings for the system (1.1) with additional prescriptions q on $\Gamma_C = \{x \in \Omega \mid x_1 = 0\}$.

Depending on the prescriptions on Γ_C relaxed formulations are obtained by transferring the original problem to optimal control problems.

1.) $\partial_n u = q$ *on* Γ_C*, measurement* $\partial_n u$ *on* Γ_O

$$J(u, q) \to \min!\,, \qquad J(w, \tau) := \frac{1}{2}\|\partial_n w - f\|^2_{\Gamma_O} \tag{2.6}$$

under the PDE-constraint

$$\begin{aligned} -\Delta u &= 0 && \text{on } \Omega\,, \\ \partial_n u &= 0 && \text{on } \Gamma_N\,, \\ u &= 0 && \text{on } \Gamma_O\,, \\ \partial_n u &= q && \text{on } \Gamma_C\,. \end{aligned} \tag{2.7}$$

2.) $u = q$ *on* Γ_C*, measurement* $\partial_n u$ *on* Γ_O

$$J(u, q) \to \min!\,, \qquad J(w, \tau) := \frac{1}{2}\|\partial_n w - f\|^2_{\Gamma_O} \tag{2.8}$$

under the PDE-constraint

$$\begin{aligned} -\Delta u &= 0 && \text{on } \Omega\,, \\ \partial_n u &= 0 && \text{on } \Gamma_N\,, \\ u &= 0 && \text{on } \Gamma_O\,, \\ u &= q && \text{on } \Gamma_C\,. \end{aligned} \tag{2.9}$$

As a third alternative we consider $\partial_n u = f$ on Γ_O as a boundary condition and interpret $u = 0$ on Γ_O as measurement.

3.) *$u = q$ on Γ_C, measurement u on Γ_O*

$$J(u,q) \to \min!\,, \qquad J(w,\tau) := \frac{1}{2}\|w\|^2_{\Gamma_O} \tag{2.10}$$

under the PDE-constraint

$$\begin{aligned} -\Delta u &= 0 && \text{on } \ \Omega\,,\\ \partial_n u &= 0 && \text{on } \ \Gamma_N\,,\\ \partial_n u &= f && \text{on } \ \Gamma_O\,,\\ u &= q && \text{on } \ \Gamma_C\,. \end{aligned} \tag{2.11}$$

Above $J(w,\tau)$ defines the so-called cost functional and q denotes a control variable.

Following Lions [4] it can be shown that these problems are uniquely solvable and that the classical regularity theory for elliptic equations applies.

3 Forward problems

In order to prepare an adequate mathematical framework for the whole problem, we first focus on a suitable formulation of the "forward" problems (2.7), (2.9) and (2.11).

3.1 Neumann-Control

A standard weak formulation is given by

$$(\nabla u, \nabla\varphi) = (g,\varphi) + \int_{\Gamma_C} q\varphi \, d\Gamma \qquad \forall \varphi \in V_q := \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma_O\} \tag{3.12}$$

and determines the solution $u \in V_q$ and may serve as a basis for applying FE methods to (2.7). Here and in what follows, $(.,.)$ represents the L^2-inner product of a bounded domain Ω in $\mathbb{R}^2$ and $\|.\|$ the corresponding norm. Furthermore, $H^m = H^m(\Omega)$ denotes the standard Sobolev space of L^2-functions with derivatives in $L^2(\Omega)$ up to the order m, and $H^1_0 \subset H^1$ is the subspace of H^1-functions vanishing on $\Gamma := \partial\Omega$.

We will apply the finite element method on decompositions $\mathbb{T}_h = \{T_i \mid 1 \le i \le N_h\}$ of Ω consisting of N_h triangular elements T_i, satisfying the usual condition of shape regularity. For ease of mesh refinement and coarsening, hanging nodes are allowed in our implementation. The width of the mesh $\mathbb{T}_h$ is characterised in terms of a piecewise constant mesh size function $h = h(x) > 0$, where $h_T := h_{|T} = \text{diam}(T)$. Using this notation, the solution $u \in V_q$ is approximated by $u_h \in V_h \subset V_q$ through

$$a(u_h,\varphi) = b(q,\varphi) \qquad \forall \varphi \in V_h\,, \tag{3.13}$$

where $a(u,\varphi) := (\nabla u, \nabla\varphi)$, $b(q,\varphi) := (g,\varphi) + \int_{\Gamma_C} q\varphi \, d\Gamma$ and V_h is a finite element space on $\mathbb{T}_h$. In this note we use standard linear finite elements to construct the spaces V_h.
The numerical results, partly taken from Dücker [2], presented throughout this work are obtained by FE-implementations based on the DEAL-library [1, 8].

3.2 Dirichlet-Control

A standard weak formulation is given by

$$(\nabla u, \nabla\varphi) = 0 \qquad \forall \varphi \in V_q := \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma_O\,,\ \varphi = q \text{ on } \Gamma_C\}\,, \tag{3.14}$$

and determines the solution $u \in V_q$ and may serve as a basis for applying FE methods to (2.9). But in view of the boundary data q being a function which has to be determined within the optimisation problem, we prefer q to appear more directly and not being hidden in the underlying space. To this end we introduce the setting

$$a(w,\varphi) := (\nabla w, \nabla\varphi) - < \partial_n w, \varphi >_0 - < w, \partial_n\varphi >_0 + \gamma < w, \varphi >_0\,, \tag{3.15}$$
$$b(q,\varphi) := \gamma(q,\varphi)_{\Gamma_C} - (q, \partial_n\varphi)_{\Gamma_C}\,, \tag{3.16}$$

where $< w, \varphi >_0 = (w,\varphi)_{\Gamma_C \cup \Gamma_O}$ and $\gamma > 0$, and employ Nitsche's method [7], which reads

$$a(u,\varphi) = b(q,\varphi)\,, \tag{3.17}$$

determining the solution $u \in V := H^1(\Omega)$.

Following the analysis presented in Nitsche [7], one chooses $\gamma = \emptyset(h^{-1})$ to obtain a FE-scheme with optimal convergence behaviour.

3.2.1 Numerical Analysis

In this subsection we recall the error analysis sketched in Nitsche [7] in a slighty more detailed way.

By using Green's formula we can show consistency with the original problem. With the standard inverse estimate (for a proof see Thomee [9])

$$\|\partial_n v\|_{0,\Gamma}^2 \leq c^2 h^{-1} |v|_{1,\Omega}^2 \;\; \forall\, v \in V_h \tag{3.18}$$

we can show that

$$a_h(u_h,u_h) \geq \frac{1}{2}|u_h|_{1,\Omega}^2 + (\psi(h) - 2c^2h^{-1})\|u_h\|_{0,\Gamma}^2.$$

By this $a_h(.,.)$ is a positive definite bilinearform in the case that $\psi(h) \geq 2c^2h^{-1}$. If we choose $\psi(h) = \gamma h^{-1}$ with $\gamma > 2c^2$ and c from (3.18) this guaranties stability and the optimal convergence behaviour can be shown next.

For the following error analysis we need the appropriate estimates for $u - I_h u$ on Ω and Γ with $k = 0, 1$:

$$\begin{aligned} \|u - I_h u\|_{k,\Omega} &\leq ch^{2-k}\|u\|_{2,\Omega} \\ \|u - I_h u\|_{k,\Gamma} &\leq ch^{3/2-k}\|u\|_{2,\Omega} \end{aligned} \tag{3.19}$$

As we can write

$$a(e,e) = \|\nabla e\|^2_{0,\Omega} - (\partial_n e, e)_{0,\Gamma} - (e, \partial_n e)_{0,\Gamma} + \gamma h^{-1}\|e\|^2_{0,\Gamma} \tag{3.20}$$

we are able to estimate $a(e,e)$ in both directions and can evaluate $|e|_{1,\Omega}$ and $\|e\|_{0,\Gamma}$.

We start with the lower bound. By using (3.19) we have

$$|(e, \partial_n e)| \leq c\|e\|_{0,\Gamma}(ch^{1/2}\|u\|_{2,\Omega} + h^{-1/2}|I_h u - u_h|_{1,\Omega}).$$

With

$$|I_h u - u_h|_{1,\Omega} \leq ch\|u\|_{2,\Omega} + |e|_{1,\Omega}$$

this results in

$$(e, \partial_n e)_{0,\Gamma} \leq \frac{1}{4}|e|^2_{1,\Omega} + ch^{-1}\|e\|^2_{0,\Gamma} + ch^2\|u\|^2_{2,\Omega}.$$

After all we have found a lower bound for $a(e,e)$:

$$a(e,e) \geq \frac{1}{2}|e|^2_{1,\Omega} + (\gamma - 2c)h^{-1}\|e\|^2_{0,\Gamma} - 2ch^2\|u\|^2_{2,\Omega}$$

For the calculation of the upper bound of (3.20) we use (3.19) and Galerkin orthogonality which results in:

$$\begin{aligned} a(e,e) \leq\ & |e|_{1,\Omega}|u - I_h u|_{1,\Omega} + \|\partial_n e\|_{0,\Gamma}\|u - I_h u\|_{0,\Gamma} + \\ & + \|\partial_n(u - I_h u)\|_{0,\Gamma}\|e\|_{0,\Gamma} + \gamma h^{-1}\|e\|_{0,\Gamma}\|u - I_h u\|_{0,\Gamma} \end{aligned} \tag{3.21}$$

Now, we can estimate each of the summands:

$$\begin{aligned} |e|_{1,\Omega}|u - I_h u|_{1,\Omega} &\leq c|e|^2_{1,\Omega} + ch^2\|u\|^2_{2,\Omega}, \\ \|\partial_n e\|_{0,\Gamma}\|u - I_h u\|_{0,\Gamma} &\leq ch^2\|u\|^2_{2,\Omega} + c|e|^2_{1,\Omega}, \\ \|\partial_n(u - I_h u)\|_{0,\Gamma}\|e\|_{0,\Gamma} &\leq ch^{-1}\|e\|^2_{0,\Gamma} + ch^2\|u\|^2_{2,\Omega}, \\ \gamma h^{-1}\|e\|_{0,\Gamma}\|u - I_h u\|_{0,\Gamma} &\leq c\gamma h^{-1}\|e\|^2_{0,\Gamma} + ch^2\|u\|^2_{2,\Omega}. \end{aligned}$$

With these four inequalities we are able to determine the upper bound for $a(e,e)$ and together with the lower bound we have

$$\begin{aligned} c|e|^2_{1,\Omega} + ch^2\|u\|^2_{2,\Omega} + c\gamma h^{-1}\|e\|^2_{0,\Gamma} &\geq a(e,e) \\ &\geq \frac{1}{2}|e|^2_{1,\Omega} + (\gamma - 2c)h^{-1}\|e\|^2_{0,\Gamma} - 2ch^2\|u\|^2_{2,\Omega} \end{aligned}$$

which leads to

$$ch^2\|u\|^2_{2,\Omega} \geq c|e|^2_{1,\Omega} + ch^{-1}\|e\|^2_{0,\Gamma}.$$

And at last we achieve the error-estimates:

$$\|e\|_{0,\Gamma} \leq ch^{3/2}\|u\|_{2,\Omega} \quad (3.22)$$

$$\|e\|_{1,\Omega} \leq ch\|u\|_{2,\Omega} \quad (3.23)$$

Considering the dual problem

$$(\nabla e, \nabla v)_{0,\Omega} - (\partial_n e, v)_{0,\Gamma} - (e, \partial_n v)_{0,\Gamma} + \gamma h^{-1}(e,v)_{0,\Gamma} = 0, \quad v \in V_h$$

and choosing $v = I_h w$ with w from

$$\begin{aligned} -\Delta w &= e \quad \text{on } \Omega \\ w &= 0 \quad \text{on } \Gamma \end{aligned} \quad (3.24)$$

leads to the L^2-error estimate with the expected optimal convergence order:

$$\|e\|_{0,\Omega} \leq ch^2\|u\|_{2,\Omega}$$

In comparison with the classical variational formulation we achieve the same optimal convergence order, the Dirichlet boundary condition isn't hidden in the underlying function space but we have to determine the constant γ which depends on the geometry of the choosen element and the degree of the polynomials we use for the approximation (see Hansbo [3]).

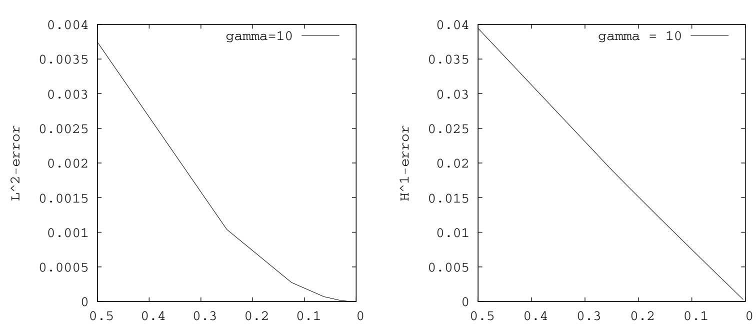

Figure 3: L^2- (left) and H^1-error (right) for the test example with $\gamma = 10$.

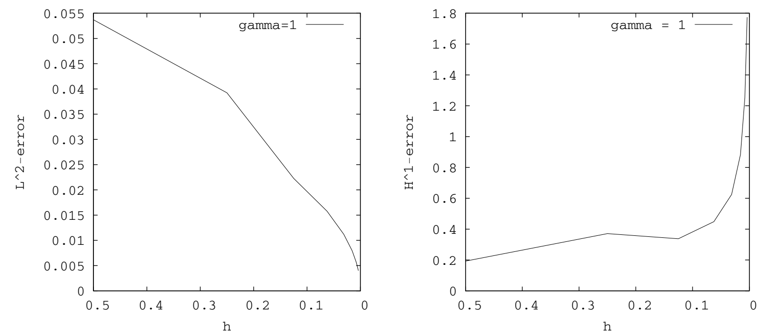

Figure 4: L^2- (left) and H^1-error (right) for the test example with $\gamma = 1$.

3.2.2 Results

As mentioned above, numerical stability of the method relies on a suitable choice of the parameter γ. We illustrate this at the model problem

$$\begin{aligned} -\Delta u &= x - x^2 + y - y^2 \quad \text{on } \Omega = (0,1)^2 \\ u &= 0 \quad \text{on } \Gamma \end{aligned} \tag{3.25}$$

with the known analytic solution $u(x, y) = \frac{1}{2}(x - y)(1 - x)(1 - y)$.

For the discretisation we choose quadrilaterals and we analyse the calculations with $\gamma = 1$ and $\gamma = 10$. Figure 3 shows the optimal convergence order for $L^2(\Omega)$- and $H^1(\Omega)$-error in the case of choosing $\gamma = 10$. In the case of $\gamma = 1$, see Figure 4, we observe the increasing of the $H^1(\Omega)$-error and the not optimal decreasing of the $L^2(\Omega)$-error which shows that this choice of γ is too small. Consequently we choose $\gamma = 10$ in our numerical experiments presented in the following sections.

4 Optimal control problems

As we have mentioned, there are three optimal control problems under consideration. They are stated in (6)(7), (8)(9) and (10)(11). Here, we first present their formal treatment.

The special focus lies on the calculation of the searched control q on Γ_C. The discretisation of q is performed by linear ansatz functions on subintervals, whose size H can be choosen independently to the mesh parameter h of the FE-space V_h.

After discretisation we can write the problems as

$$J(u_h, q_H) \rightarrow \min \tag{4.26}$$

$$Au_h = g + Bq_H, \tag{4.27}$$

where A, B, g correspond to the forms $a(.,.), b(.,)$ and $(g,.)$, respectively. u_h and q_H represent the related solutions.
Now, if A is regular u_h can be written as

$$u_h = A^{-1}g + A^{-1}Bq_H. \tag{4.28}$$

From the theory of inverse problems we know that the minimisation problem in (4.26),(4.27) can be rewritten in the regularised form

$$(B^T A^{-T} C A^{-1} B + \alpha R^T R) q_H = B^T A^{-T}(Cf - CA^{-1}g). \tag{4.29}$$

where the positive definite matrix C is formally employed to describe the evaluation of $J(.,.)$,
α defines a regularisation parameter and R denotes the regularisation, e.g. the choice $R = Id$ yields the classical Tikhonov-Philips regularisation.

The discrete problems are solved by using a pcg-method with multigrid-steps as preconditioner yielding a solution for q. In a postprocess problem relevant data (e.g. u_h) are determined. For technical and algorithmical details we refer to Dücker [2].

Our techniques are tested at the following model problem. We consider

$$-\Delta u = g \tag{4.30}$$

with $g = 12xy^2 - 12xy + 2x - 12y^2 + 12y - 2$ and the known analytic solution

$$u(x,y) = y^2(1-y)^2(1-x)$$

with normal derivatives

$$\partial_n u|_{\Gamma_C} = y^2(1-y)^2, \quad \partial_n u|_{\Gamma_O} = -y^2(1-y)^2.$$

We have analysed the results for different global grid refinements, different number of DOF on Γ_C and different regularisation parameters.

For the choice R we employ three types of regularisation, corresponding to discrete derivatives of order $0, 1, 2$, i.e. we choose $R_H^0 = Id$,

$$R_H^1 = \frac{1}{H}\begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & -1 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 \end{pmatrix}, \quad R_H^2 = \frac{1}{H^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & -1 & 2 & -1 & 0 \\ 0 & \ddots & \ddots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix},$$

where we assume the degrees of freedom along Γ_C to be numbered canonically.

In the following we present the numerical results of the three problems.

4.1 Optimal Neumann-control problem, Neumann measurement

In this case we want to solve (2.6) with the underlying PDE-constraint (2.7).

Figure 5 shows the results of the searched control $\partial_n u|_{\Gamma_C}$ calculated with the three regularisation matrices with reduced number of DOF on Γ_C and the corresponding regularisation parameter. For all three calculations we reach good results. In the case of regularising with an approximation of the second derivative we can use the smallest number of DOF on Γ_C.

With this calculated control we also reach good results for $u|_{\Gamma_C}$, $\partial_n u|_{\Gamma_O}$ and $u|_{\Gamma_O}$ which we don't want to present here.

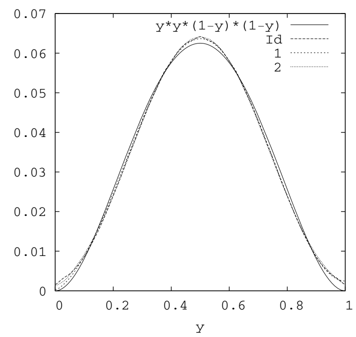

Figure 5: $\partial_n u|_{\Gamma_C} = y^2(1-y)^2$ and $\partial_n u_h|_{\Gamma_C}$ refining the grid eight times. I: $R = Id$, 41 DOF on Γ_C and $\alpha = 1.e-09$, 1: $R = R_H^1$, 91 DOF on Γ_C and $\alpha = 1.e-12$, 2: $R = R_H^2$, 31 DOF on Γ_C and $\alpha = 1.e-12$

4.2 Optimal Dirichlet-control problem, Neumann measurement

In the case of searched Dirichlet control for given Neumann measurement (see equation (2.8) and the PDE-constraint (2.9)) we also reach good results for the searched control on Γ_C (see Figure 6 left). But in this case we have problems in calculating $\partial_n u_h|_{\Gamma_C}$ with the computed control in the case of regularising with the Identity. There we achieve oscillations which we are not able to avoid. If we use the approximation of first or second derivative as regularisation we also achieve oscillations but we are able to avoid them by enlarging the number of DOF on Γ_C in comparison to the case of searched Neumann control and given Neumann measurement (see Figure 6 right).

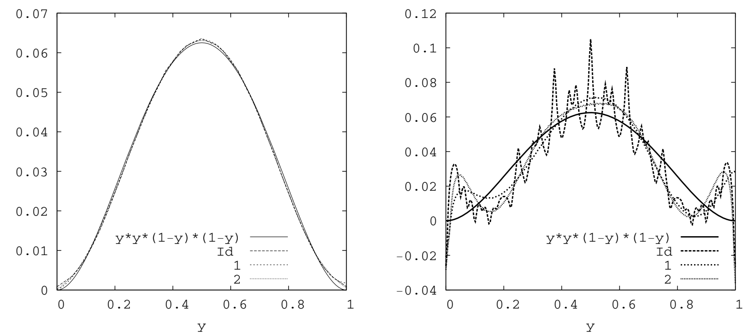

Figure 6: $u|_{\Gamma_C} = y^2(1-y)^2$ and $u_h|_{\Gamma_C}$ (left) and $\partial_n u|_{\Gamma_C} = y^2(1-y)^2$ and $\partial_n u_h|_{\Gamma_C}$ (right) refining eight times calculated with $R = Id$, 41 DOF on Γ_C and $\alpha = 1.e-11$; $R = R_H^1$ (1), 101 DOF on Γ_C and $\alpha = 1.e-14$ and $R = R_H^2$ (2), 91 DOF on Γ_C and $\alpha = 1.e-17$.

4.3 Optimal Dirichlet-control problem, Dirichlet measurement

In this case (see equation (2.10) and the PDE-constraint (2.11)) we have similar results as for searched Dirichlet control and given Neumann measurement. The results for the searched control are good for all regularisations but in the calculation of $\partial_n u_h|_{\Gamma_C}$ we achieve oscillations (see Figure 7) which especially in the case of the classical Tikhonov regularisation could not be eliminated.

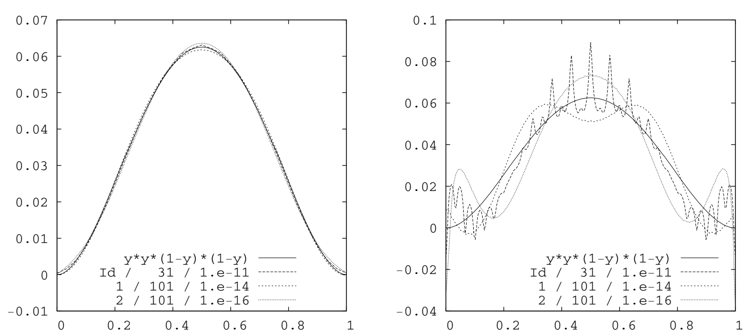

Figure 7: $u|_{\Gamma_C} = y^2(1-y)^2$ and $u_h|_{\Gamma_C}$ (left) and $\partial_n u|_{\Gamma_C} = y^2(1-y)^2$ and $\partial_n u_h|_{\Gamma_C}$ (right) for refining the grid eight times calculated with $R = Id$, 31 DOF on Γ_C and $\alpha = 1.e-11$; $R = R_H^1$ (1), 101 DOF on Γ_C and $\alpha = 1.e-14$; $R = R_H^2$ (2), 101 DOF on Γ_C and $\alpha = 1.e-16$

Summarising the above: The best results are obtained for searched Neumann control and given Neumann measurement. The more Dirichlet data, the worse the results.

5 APPLICATION: ELKRAFT-PROBLEM

Based on the results of the model problem and given Neumann measurements on Γ_O in the application of hybrid insulation we consider the case of searched Neumann control on Γ_C. The used data are from the NTNU Trondheim.

For controlling the results in the cases with unknown solution, we produce a reference solution by solving the forward problem with prescribed control as given boundary data.

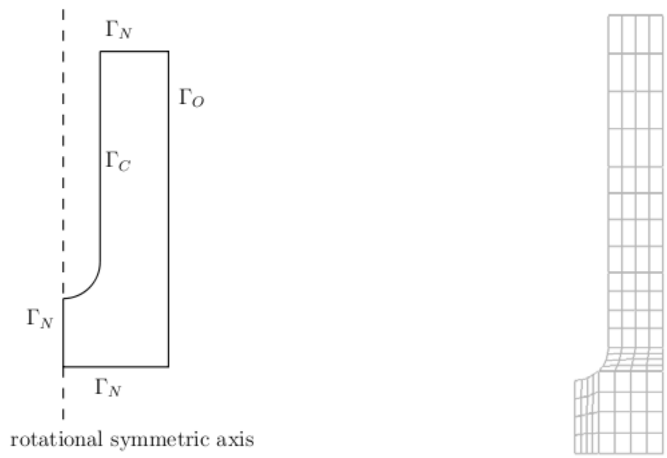

Figure 8: Sketch of geometry and boundary conditions (left) and the corresponding coarse FE-mesh (right).

Figure 9 shows the searched control $\partial_n u_h|_{\Gamma_C}$ (left) and on the right the given measurement and $\partial_n u_h|_{\Gamma_O}$ computed with the calculated control.

Figure 10 shows a video image from the experimental setup (from F. Mauseth NTNU Trondheim) and the calculated u on the rotational symmetric geometry for the given measurement.

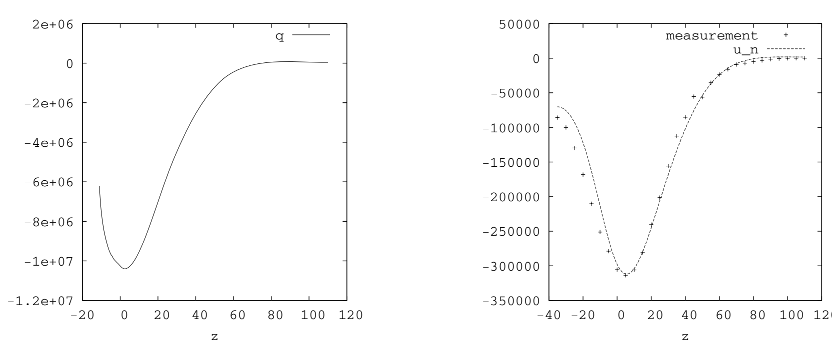

Figure 9: The calculated control $q = \partial_n u_h|_{\Gamma_C}$ (left), the given measurements and the calculated normal derivative $\partial_n u_h|_{\Gamma_O}$ (right).

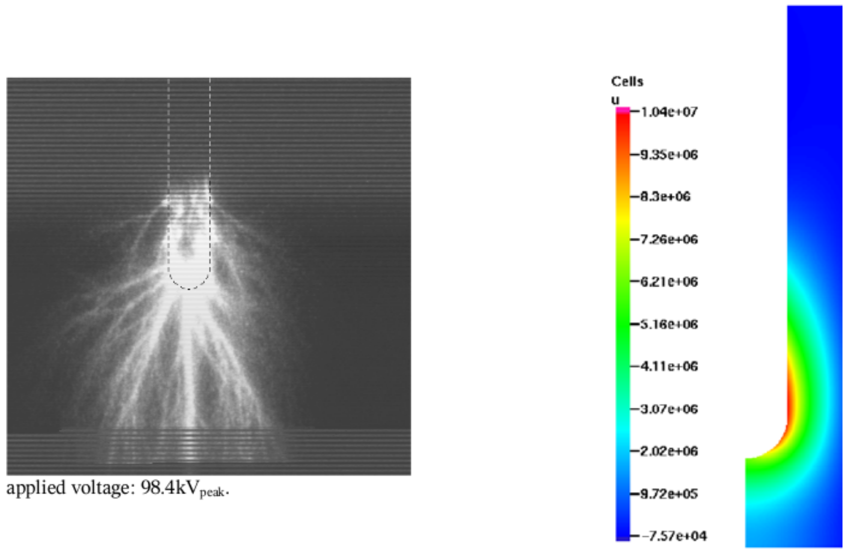

Figure 10: Video image (left, from F. Mauseth NTNU Trondheim) and calculated u_h on Ω for the given measurements $\partial_n u|_{\Gamma_O}$ (illustrated in Figure 9 right).

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