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An $M^X/(G_1,G_2)/1$ Vacation Queue with Balking and Optional Re-service

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Abstract

The present paper aims at studying a queuing model with two stage heterogeneous service where customer arrival is in batches and has a single server providing service in two stages, one after the other in succession. A service is said to be completed when both services are done. Also the server is assumed to go for vacation after completion of the two stages of service with probability *p* or may continue serving the next customer with probability *(1-p)*. In addition, we have included balking which reflects customer's impatient behavior where an arriving batch decides not to join the queue for some reasons. We also added optional re-service where a customer may require re-service for any of the services or depart from the system once service is completed. We derive the steady state queue size distribution and discuss some special cases.

Keywords: Queue size, Batch Arrival, Two-Stage Heterogeneous Services, Balking, Optional Re-service

Introduction

Queues with server vacations have gained considerable importance in the area of research in recent years. It has emerged as an important area of study and researchers has been exploring different concepts of queuing models with vacations. Most of the literature in queuing with batch arrival and vacation has been studied by eminent authors like Lee et al [9],Takagi [8], Shanthikumar, J.G [11] Choudhury, G. & Madan, K.C.[6], Keilson and Servi [10], to quote a few. A major contribution in studies on two stage batch arrival queuing system is seen by authors like Choudhury, G and Madan, K.C. [7]. In the last few years, we see studies on queues with balking and reneging gaining significant importance. Queuing system with impatient customers is a familiar phenomenon we come across in many real life situations. Customers may be discouraged to join a queue due to long waiting time or service times or for other constraints and may leave the queue without joining. This is balking. We see applications of queue with balking in emergency services in hospitals dealing serious patients, communication systems, production and inventory system and many more.

A queue with balking was initially studied by Haight [5]. Since then, extensive amount of work has been done on queuing systems related to impatient customers. Queues with balking has been studied by authors like Altman, E. and Yechiali, U. [4] ,Ancker et al [2], Choudhury, A. and Medhi, P [1], Zhang et al [18] in the last few years. Most of the authors mentioned here have studied single server models.

Re-service is also a real life phenomenon where customers receiving some kind of service may need to repeat or demand re-service for the service taken. Patients might demand re-service while visiting a doctor's clinic, manufacturing units in an industry while inspection may spot some defects and as a result may require re-service and many real life applications. Re-service was initially studied by Madan [15].Recently the concept of re-service has been studied by authors like Tadj and Ke [16], Jayakumar and Arumuganathan [17]. Though we see few works on re-service, it is an important aspect of queuing theory and can be studied more extensively.

Here we study the steady state behavior of a bulk queue system having two-stage general heterogeneous service along with balking and re-service with server vacations. We may thus denote our model as $M^X/(G_1, G_2)/V_s/BL/OR$.

We define the mathematical model under the assumptions given below.

1. The Mathematical Model

Customers arrive in batches following a compound Poisson Process with rate of arrival λ.

Let $\lambda a_i\, dt$ (i = 1, 2, 3….) be the first order probability that customers in batches of size *i* arrive at the system at a short interval of time (x, x+dt], where $0 \le a_i \le 1$ and $\sum_{i=1}^{\infty} a_i = 1$

The server provides two stages of heterogeneous services one after the other. An arrival batch shall first receive the service offered by server 1 followed by the service offered by second server, defined as the first stage (FS) and second stage (SS) service respectively. The service discipline is assumed to be on a first come first served basis (FCFS).

We assume that the service time S_j (j =1,2) of the j th stage service follows a general probability distribution with distribution function $B_j(s_j)$, $b_j(s_j)$ being the probability density function and $E(S_j^n)$ as the n th moment of the service time, j=1,2

Let $\mu_j(x)$ be the conditional probability of stage j service during the period (x, x+dt] given elapsed time is *x* such that

$$\mu_j(x) = \frac{b_j(x)}{1 - B_j(x)} \quad j = 1,2 \tag{1}$$

and $$b_j(s_j) = \mu_j(x) \exp\left[-\int_0^s \mu_j(x)dx \right], \; j = 1,2 \tag{2}$$

Once the second stage service (SS) of a unit is complete the server is assumed to take vacation with probability *p* or may continue to offer service with probability (*1-p)*. As soon as the vacation period of the server is over, he joins the system to continue service of the waiting customers.

We assume the vacation time to be a random variable following general probability law with distribution function given by $W(v)$ and density function by *w(v)* and $E(v^n)$ is the *n*th moment (n =1,2,….)

Here we assume that $\phi(x)$ be the conditional probability of a vacation period during the interval (x, x+dx] , given that elapsed time is x, so that

$$\phi(x) = \frac{W(x)}{1 - W(x)} \tag{3}$$

and thus

$$w(v) = \phi(v) \exp\left[-\int_0^v \phi(x) dx \right] \tag{4}$$

In addition, we assume that customers arriving for service may become impatient by observing the queue size or the server being busy, may balk (refuse to join the system). Here we assume that b_1 is the probability that arriving batch joins the system at the time when server is busy and b_2 is the probability that an arriving batch joins the system during the period when server is on vacation.

Also we assume that as soon as second stage service is complete the customer has the choice to leave the system or join the system again for re-service, if necessary. A customer may repeat the service in the two stages with probability r or may leave the system with probability $(1 - r)$.

3. Definitions and Notations

We assume that steady state exists and define

$P_{n,j}(x) =$ Probability that there are *n (≥ 1)* customers in the system including

one customer in type *j* service , *j =1,2* and elapsed service time is *x*

Thus $P_{n,j} = \int_0^\infty P_{n,j}(x) dx$ is the corresponding steady state probability irrespective of elapsed time *x*.

$R_{n,j}(x) =$ Probability that there are *n (≥ 1)* customers in the system including one customer who is repeating service, *j =1, 2.* and elapsed service time is *x*.

Accordingly $R_{n,j} = \int_0^\infty R_{n,j}(x)dx$ is the corresponding steady state probability irrespective of elapsed service time *x*.

$V_n(x)$ = probability that there are *n (≥ 0)* customers in the queue and server is on vacation and elapsed vacation time is *x*. $V_n = \int_0^\infty V_n(x)dx$ is the corresponding steady state probability irrespective of elapsed vacation time *x*.

Q = Steady state probability of the server is idle as the server takes vacation

The Probability Generating Functions are defined as:

$$P_j(x,z) = \sum_{n=1}^{\infty} z^n P_{n,j}(x),\ P_j(z) = \sum_{n=1}^{\infty} z^n P_{n,j}\ ,\ |z| \le 1 \quad j = 1,2\,, \tag{5}$$

$$R_j(x,z) = \sum_{n=1}^{\infty} z^n R_{n,j}(x)\,,\ R_j(z) = \sum_{n=1}^{\infty} z^n R_{n,j}\ ;\ |z| \le 1\ \ j = 1,2 \tag{6}$$

$$V(x,z) = \sum_{n=0}^{\infty} z^n V_n(x), \qquad V(z) = \sum_{n=1}^{\infty} z^n V_n \quad ;\ |z| \le 1\ \ j = 1,2\,; \tag{7}$$

$$A(z) = \sum_{i=1}^{\infty} z^i a_i \ ;\ |z| \le 1$$

4. Equations governing the system

Under this model we construct the differential equations as

$$\frac{d}{dx}P_{n,1}(x) + \{\lambda + \mu_1(x)\}P_{n,1}(x) = \lambda(1-b_1)P_{n,1}(x) + \lambda\sum_{i=1}^{n} a_i P_{n-i,1}(x) \quad ;\ n \ge 1 \tag{8}$$

$$\frac{d}{dx}P_{n,2}(x) + \{\lambda + \mu_2(x)\}P_{n,2}(x) = \lambda(1-b_1)P_{n,2}(x) + \lambda\sum_{i=1}^{n} a_i P_{n-i,2}(x) \ ;\ n \ge 1 \tag{9}$$

$$\frac{d}{dx}R_{n,1}(x)+\{\lambda+\mu_1(x)\}R_{n,1}(x)=\lambda(1-b_1)R_{n,1}(x)+\lambda\sum_{i=1}^{\infty}a_iR_{n-i,i}(x)\quad;\ n\geq 1 \tag{10}$$

$$\frac{d}{dx}R_{n,2}(x)+\{\lambda+\mu_2(x)\}R_{n,2}(x)=\lambda(1-b_1)R_{n,2}(x)+\lambda\sum_{i-1}^{\infty}a_iR_{n-i,2}(x)\ ;\ \ n\geq 1 \tag{11}$$

$$\frac{d}{dx}V_n(x)+\{\lambda+\phi(x)\}V_n(x)=\lambda(1-b_2)+b_2\lambda\sum_{i=1}^{n}a_iV_{n-i}(x)\quad;\qquad n\geq 0 \tag{12}$$

$$\frac{d}{dx}V_0(x)+\{\lambda+\phi(x)\}V_0(x)=\lambda(1-b_2)V_0(x) \tag{13}$$

$$\lambda Q=\lambda(1-b_1)Q+(1-p)(1-r)\int_0^{\infty}P_{1,2}(x)\mu_1(x)dx+\int_0^{\infty}V_0(x)\phi(x)dx \tag{14}$$

where $P_{0,j}(x)=0, R_{0,j}(x)=0$, $j=1,2$ occurring in the equations (8)-(11) given above.

The above differential equations now have to be solved subject to the following boundary conditions:

$$\begin{aligned}P_{n,1}(0)&=\lambda b_1\,a_nQ+(1-p)(1-r)\int_0^{\infty}P_{n+1,2}(x)\mu_2(x)dx\\&\quad+(1-p)\int_0^{\infty}R_{n+1,2}(x)\mu_2(x)x+\int_0^{\infty}V_n(x)\phi(x)dx\qquad n\geq 1\end{aligned} \tag{15}$$

$$P_{n,2}(0)=(1-r)\int_0^{\infty}P_{n,1}(x)\mu_1(x)dx+\int_0^{\infty}R_{n,1}(x)\mu_1(x)dx\qquad n\geq 1 \tag{16}$$

$$R_{n,1}(0)=r\int_0^{\infty}P_{n,1}(x)\mu_1(x)dx\qquad n\geq 1 \tag{17}$$

$$R_{n,2}(0)=r\int_0^{\infty}P_{n,2}(x)\mu_2(x)dx\qquad n\geq 1 \tag{18}$$

$$V_n(0)=p(1-r)\int_0^{\infty}P_{n+1,2}(x)\mu_2(x)dx\qquad n\geq 0 \tag{19}$$

5. Steady State Queue distribution

Multiplying equations (8)-(12) by z^n , summing over suitable values of *n* and simplifying we get

$$\frac{d}{dx}P_1(x,z) + \{b_1(\lambda - \lambda A(z)) + \mu_1(x)\}P_1(x,z) = 0 \tag{20}$$

$$\frac{d}{dx}P_2(x,z) + \{b_1(\lambda - \lambda A(z)) + \mu_2(x)\}P_2(x,z) = 0 \tag{21}$$

$$\frac{d}{dx}R_1(x,z) + \{b_1(\lambda - \lambda A(z)) + \mu_1(x)\}R_1(x,z) = 0 \tag{22}$$

$$\frac{d}{dx}R_2(x,z) + \{b_1(\lambda - \lambda A(z)) + \mu_2(x)\}R_2(x,z) = 0 \tag{23}$$

$$\frac{d}{dx}V(x,z) + \{b_2(\lambda - \lambda A(z)) + \phi(x)\}V(x,z) = 0 \tag{24}$$

Integrating equations (19)-(23) between limits 0 to *x* we obtain

$$P_1(x,z) = P_1(0,z)\exp[-b_1(\lambda - \lambda A(z))x - \int_0^\infty \mu_1(t)dt\,] \tag{25}$$

$$P_2(x,z) = P_2(0,z)\exp[-b_1(\lambda - \lambda A(z))x - \int_0^\infty \mu_2(t)dt\;\;] \tag{26}$$

$$(26)\; R_1(x,z) = R_1(0,z)\exp[-b_1(\lambda - \lambda A(z))x - \int_0^\infty \mu_1(t)dt\;\;] \tag{27}$$

$$R_2(x,z) = R_2(0,z)\exp[-b_1(\lambda - \lambda A(z))x - \int_0^\infty \mu_2(t)dt\;\;] \tag{28}$$

$$V(x,z) = V(0,z)\exp[-b_2(\lambda - \lambda(z))x - \int_0^\infty \phi(t)dt\,] \tag{29}$$

Equations (25)-(29) hold for all $x > 0$

We next multiply the boundary conditions by suitable powers of z^n and taking summation over all possible values of *n* and using (14) we get after simplification

$$zP_1(0,z) = z\lambda b_1(A(z)-1)Q + (1-p)(1-r)\int_0^\infty P_2(x,z)\mu_2(x)dx + (1-p)\int_0^\infty R_2(x,z)\mu_2(x)dx + z\int_0^\infty V(x,z)\phi(x)dx \tag{30}$$

$$P_2(0,z) = (1-r)\int_0^\infty P_1(x,z)\mu_1(x)dx + \int_0^\infty R_1(x,z)\mu_1(x)dx \tag{31}$$

$$R_1(0,z) = r\int_0^\infty P_1(x,z)\mu_1(x)dx \tag{32}$$

$$R_2(0,z) = r\int_0^\infty P_2(x,z)\mu_2(x)dx \tag{33}$$

$$zV(0,z) = p(1-r)\int_0^\infty P_2(x,z)\mu_2(x)dx \tag{34}$$

Again multiplying equation (25) by $\mu_1(x)$ and integrating by parts w.r.t. *x*, between the limits 0 to ∞, we obtain

$$\int_0^\infty P_1(x,z)\mu_1(x)dx = P_1(0,z)B_1^*(b_1(\lambda-\lambda A(z))) \tag{35}$$

Proceeding similarly for equations (25)-(28), we obtain

$$\int_0^\infty P_2(x,z)\mu_2(x)dx = P_2(0,z)B_2^*(b_1(\lambda-\lambda A(z))) \tag{36}$$

$$\int_0^\infty R_1(x,z)\mu_1(x)dx = R_1(0,z)B_1^*(b_1(\lambda-\lambda A(z))) \tag{37}$$

$$\int_0^\infty R_2(x,z)\mu_2(x)dx = R_2(0,z)B_2^*(b_1(\lambda-\lambda A(z))) \tag{38}$$

$$\int_0^\infty V(x,z)\phi(x)x = V(0,z)W^*(b_2(\lambda-\lambda A(z))) \tag{39}$$

Now utilizing relations (35)-(39) in equations (30)-(34) we get

$$\begin{aligned} zP_1(0,z) &= zb_1\lambda(A(z)-1)Q + (1-p)(1-r)P_2(0,z)B_2^*(b_1(\lambda-\lambda A(z))) \\ &\quad + (1-p)R_2(0,z)B_2^*(b_1(\lambda-\lambda A(z))) + zV(0,z)W^*(b_2(\lambda-\lambda A(z))) \end{aligned} \tag{40}$$

$$P_2(0,z) = (1-r)\,P_1(0,z)\,B_1^*\big(b_1(\lambda-\lambda A(z))\big) + R_1(0,z)\,B_1^*\big(b_1(\lambda-\lambda A(z))\big) \tag{41}$$

$$R_1(0,z) = rP_1(0,z)B_1^*(b_1(\lambda-\lambda A(z))) \tag{42}$$

$$\begin{aligned} R_2(0,z) &= r\,P_2(0,z)\,B_2^*(b_1(\lambda-\lambda A(z))) \\ &= r(1-r)P_1(0,z)B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z))) + rP_1(0,z)[B_1^*(b_1(\lambda-\lambda A(z)))]^2 \end{aligned} \tag{43}$$

$$\begin{aligned} zV(0,z) &= p(1-r)P_2(0,z)B_2^*(b_1(\lambda-\lambda A(z))) \\ &= p(1-r)^2\,P_1(0,z)B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z))) \\ &\quad + pr(1-r)[B_1^*(b_1(\lambda-\lambda A(z)]^2\,B_2^*(b_1(\lambda-\lambda A(z))) \end{aligned} \tag{44}$$

Where $B_j^*(b_1(\lambda-\lambda A(z))) = \int_0^\infty e^{-b_1(\lambda-\lambda A(z))x}\,dB_j(x)$ is the Laplace Transform of B_j for $j=1, 2$ and $W^*(b_2(\lambda-\lambda A(z))) = \int_0^\infty e^{-b_2(\lambda-\lambda A(z))x}\,dW(x)$ is the Laplace Transform of W.

Substituting the values from (41), (42) (43) and (44) in (40) we get

$$\begin{aligned} zP_1(0,z) &= zb_1\lambda(A(z)-1)Q + (1-p)(1-r)\begin{Bmatrix}(1-r)P_1(0,z)B_1^*(b_1[\lambda-\lambda(z)]) \\ +rP_1(0,z)B_1^*(b_1(\lambda-\lambda A(z)))\end{Bmatrix}B_2^*(b_1(\lambda-\lambda A(z))) \\ &\quad + (1-p)\{r(1-r)P_1(0,z)B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z))) \\ &\quad + rP_1(0,z)[B_1^*(b_1(\lambda-\lambda A(z)))]^2\}B_2^*(b_1(\lambda-\lambda A(z))) \\ &\quad + p(1-r)^2\,P_1(0,z)B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1[\lambda-\lambda A(z)])W^*(b_2[\lambda-\lambda(z)]) \\ &\quad + p\,r(1-r)[B_1^*(b_1(\lambda-\lambda A(z)))]^2\,B_2^*(b_1(\lambda-\lambda A(z)))\,W^*(b_2(\lambda-\lambda A(z))) \end{aligned}$$

After simplifying we get

$$P_1(0,z)=\frac{z b_1\lambda\left(A(z)-1\right)Q}{z-\left[\begin{array}{l}\left\{(1-p)+pW^*\left(b_2\left(\lambda-\lambda A(z)\right)\right)\right\}\left\{\begin{array}{l}(1-r)^2B_1^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)B_2^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)+\\ r(1-r)\left[B_1^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)\right]^2B_2^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)\end{array}\right\}\\ +r(1-p)B_1^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)B_2^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)\left\{\begin{array}{l}(1-r)B_2^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)\\ +rB_1^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)B_2^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)\end{array}\right\}\end{array}\right]} \tag{45}$$

Again substituting $P_1(0,z)$ from (44) in (43) we obtain

$$V(0,z)=\frac{\left\{p(1-r)^2+pr(1-r)B_1^*(b_1(\lambda-\lambda A(z)))\right\}z\lambda b_1(A(z)-1)QB_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))}{z-\left\{\begin{array}{l}\{(1-p)+pW^*(b_2(\lambda-\lambda A(z)))\}\left\{\begin{array}{l}(1-r)^2B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))+\\ r(1-r)[B_1^*(b_1(\lambda-\lambda A(z)))]^2B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right\}\\ +r(1-p)B_1^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)B_2^*\left(b_1\left(\lambda-\lambda A(z)\right)\right)\left\{\begin{array}{l}(1-r)B_2^*(b_1(\lambda-\lambda A(z)))\\ +rB_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right\}\end{array}\right\}} \tag{46}$$

And thus

$$P_2(0,z)=\frac{z b_1\lambda(A(z)-1)Q\left\{(1-r)B_1^*(b_1(\lambda-\lambda A(z)))+r[B_1^*(b_1(\lambda-\lambda A(z)))]^2\right\}}{z-\left[\begin{array}{l}\left\{(1-p)+pW^*(b_2(\lambda-\lambda A(z)))\right\}\left\{\begin{array}{l}(1-r)^2B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))+\\ r(1-r)[B_1^*(b_1(\lambda-\lambda A(z)))]^2B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right\}\\ +r(1-p)\left\{(1-r)B_2^*(b_1(\lambda-\lambda A(z)))+rB_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\right\}B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right]} \tag{47}$$

$$R_1(0,z)=\frac{rzb_1\lambda(A(z)-1)QB_1^*(b_1(\lambda-\lambda A(z)))}{z-\left\{\begin{array}{l}\{(1-p)+pW^*(b_2(\lambda-\lambda A(z)))\}\left\{\begin{array}{l}(1-r)^2B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\\+r(1-r)\,[B_1^*(b_1(\lambda-\lambda A(z)))]^2B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right\}\\+r(1-p)B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\left\{\begin{array}{l}(1-r)B_2^*(b_1(\lambda-\lambda A(z)))\\+rB_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right\}\end{array}\right\}} \tag{48}$$

$$R_2(0,z)=\frac{rzb_1\lambda(A(z)-1)\{(1-r)B_1^*(b_1(\lambda-\lambda A(z)))+r[B_1^*(b_1(\lambda-\lambda A(z)))]^2\}QB_2^*(b_1(\lambda-\lambda A(z)))}{z-\left\{\begin{array}{l}\{(1-p)+pW^*(b_2(\lambda-\lambda A(z)))\}\left\{\begin{array}{l}(1-r)^2B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\\+r(1-r)[B_1^*(b_1(\lambda-\lambda A(z)))]^2B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right\}\\+r(1-p)B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\left\{\begin{array}{l}(1-r)B_2^*(b_1(\lambda-\lambda A(z)))\\+rB_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))\end{array}\right\}\end{array}\right\}} \tag{49}$$

We further integrate equations (25)-(29) w.r.t x and after utilizing relation (2) obtain

$$P_1(z)=\frac{z\{1-B_1^*(b_1[\lambda-\lambda A(z)])\}Q}{D(z)} \tag{50}$$

Similarly

$$P_2(z)=\frac{z\{(1-r)B_1^*(b_1(\lambda-\lambda A(z)))+r\{B_1^*(b_1[\lambda-\lambda A(z)])\}^2\}\left[1-B_2^*(b_1[\lambda-\lambda A(z)])\right]Q}{D(z)} \tag{51}$$

$$R_1(z) = \frac{rzB_1^*(b_1[\lambda - \lambda A(z)])\left[1 - B_1^*(b_1[\lambda - \lambda A(z)])\right]Q}{D(z)} \tag{52}$$

$$R_2(z) = \frac{rzB_2^*(b_1[\lambda - \lambda A(z)])\left\{\begin{matrix}(1-r)B_1^*(b_1(\lambda - \lambda A(z)))\\ +r[B_1^*(b_1(\lambda - \lambda A(z)))]^2\end{matrix}\right\}\{1 - B_2^*(b_1[\lambda - \lambda A(z)])\}Q}{D(z)} \tag{53}$$

$$V(z) = \frac{\left\{\begin{matrix}p(1-r)^2 B_1^*(b_1[\lambda - \lambda A(z)])B_2^*(b_1[\lambda - \lambda A(z)])\\ +pr(1-r)\{B_1^*(b_1[\lambda - \lambda A(z)])\}^2 B_2^*(b_1[\lambda - \lambda A(z)])\end{matrix}\right\}\{1 - W^*(b_2[\lambda - \lambda A(z)])\}Q}{D(z)} \tag{54}$$

where

$$D(z) = \left[\begin{matrix}\{(1-p) + pW^*(b_2[\lambda - \lambda A(z)])\}\left\{\begin{matrix}(1-r)^2 B_1^*(b_1[\lambda - \lambda A(z)])B_2^*(b_1[\lambda - \lambda A(z)])\\ +r(1-r)\{B_1^*(b_1[\lambda - \lambda A(z)])\}^2 B_2^*(b_1[\lambda - \lambda A(z)])\end{matrix}\right\}\\ +r(1-p)B_1^*(b_1(\lambda - \lambda A(z)))B_2^*(b_1(\lambda - \lambda A(z)))\left\{\begin{matrix}(1-r)B_2^*(b_1[\lambda - \lambda A(z)])\\ +rB_1^*(b_1[\lambda - \lambda A(z)])B_2^*(b_1[\lambda - \lambda A(z)])\end{matrix}\right\}\end{matrix}\right] - z \tag{55}$$

Let $P_q(z)$ be the probability generating function irrespective of the type of service being provided by the server, such that we have

$$P_q(z) = P_1(z) + P_2(z) + R_1(z) + R_2(z) + V(z) = \frac{N(z)}{D(z)} \tag{56}$$

To find Q we use the normalizing condition

$$Q + P_1(1) + P_2(1) + R_2(1) + V(1) = 1$$

Since the R. H. S of (55) is indeterminate of the zero/zero form at $z = 1$, applying L'Hopital's rule we obtain

$$P_q(1)=\lim_{z\to 1}P_q(z)$$
$$=\frac{\lambda E(I)\left[b_1(1+r)\{E(B_1)+E(B_2)\}+p(1-r)b_2E(V)\right]}{1-\lambda E(I)\begin{bmatrix}b_1(1-r)^2\{E(B_1)+E(B_2)\}+p(1-r)b_2E(V)\\+2r(1-p)b_1\{E(B_1)+E(B_2)\}\\+r(1-r)\{(1-p)E(B_1)+E(B_2)\}\end{bmatrix}} \quad (57)$$

Where E(I) is the mean size of arriving customers, $E(B_1)$ is the mean of service time of first stage service, $E(B_2)$ is the mean service time of second stage service and E(V) is the mean of vacation time, $B^*(0)=1$ and $W^*(0)=1$.

Now using the relation $Q+P_q(1)=1$, we obtain

$$Q=\frac{1-\lambda E(I)\begin{bmatrix}(1+r)b_1\{E(B_1)+E(B_2)\}+p(1-r)b_2E(V)\\+b_1r\{2(1-r)+(r+1)(1-p)\}E(B_1)\\+b_1r\{(1-r)+(1-p)(2-r)\}E(B_2)\end{bmatrix}}{1-\lambda b_1rE(I)\left[\{(2-r)-p(r+1)\}E(B_1)+\{2(1-r)-p(2-r)\}E(B_2)\right]} \quad (58)$$

and $Q=1-\rho$ gives

$$\rho=\lambda E(I)\left[b_1\{1+r(1+r)\}E(B_1)+b_1(1+r)E(B_2)\right]+p(1-r)b_2E(V) \quad <1 \quad (59)$$

is the stability condition under which steady state exists.

Now let $P_s(z)=P_1(z)+P_2(z)+R_2(z)+zV(z)+Q$ be the steady state queue size distribution. Using the above relations we obtain

$$P_S(z)=\frac{\begin{array}{l}Q\Big|\ (1-z)(1-r)^2\Big[(1-p)+pW^*(b_2[\lambda-\lambda A(z)])\Big]B_1^*(b_1(\lambda-\lambda A(z)])B_2^*(b_1[\lambda-\lambda A(z)]) \\ +\{(1-p)-z\}r\left\{\begin{array}{l}r\{B_1^*(b_1[\lambda-\lambda A(z)])B_2^*(b_1[\lambda-\lambda A(z)])\}^2 \\ +r(1-r)B_1^*(b_1[\lambda-\lambda A(z)])\{B_2^*(b_1[\lambda-\lambda A(z)])\}^2\end{array}\right\} \\ +r(1-r)\{B_1^*(b_1[\lambda-\lambda a(z)])\}^2 B_2^*(b_1[\lambda-\lambda A(z)])\ \Big|\end{array}}{D(z)} \tag{60}$$

and $D(z)$ is given by equation (55)

Thus relation (60) gives the steady state queue size of an $M^X/(G_{1,}G_2)/V_s/BL/OR$ queuing system.

6. Mean Queue Size and Mean Waiting Time in Queue

Here we derive the mean queue size of this $M^X/(G_1,G_2)/V_s/BL/OR$ queue.

Let L_Q denote the mean queue size at random epoch, then

$$L_Q=\frac{dP_q(z)}{dz}\Big]_{z=1} \tag{61}$$

Using L'Hopital's Rule twice as the R. H.S of $P_q(z)$ is indeterminate of the 0/0 form

$$L_Q=\lim_{z\to 1}\frac{D^{/}(z)N^{//}(z)-N^{/}(z)D^{//}(z)}{2(D^{/}(z))^2} \tag{62}$$

where primes and double primes denote the first and second derivatives at z =1

Simplifying (56) we get

$$N^{/}(1)=Q[p(1-r)b_2\lambda E(I)E(V)+(1+r)b_1\lambda E(I)\{E(B_1)+E(B_2)\}]$$

$$N^{//}(1)=Q\begin{bmatrix}\lambda E(I/I-1)\left[b_1(1+r)\{E(B_1)+E(B_2)\}+p(1-r)b_2E(V)\right]\\-\left(\lambda E(I)\right)^2\left[b_1{}^2(1+r)\{E(B_1^2)+E(B_2^2)\}+p(1-r)b_2^2E(V^2)\right]\end{bmatrix}$$

$$D'(1)=1-\lambda E(I)\begin{bmatrix}(1-r)^2b_1\{E(B_1)+E(B_2)\}+p(1-r)b_2E(V)\\+2r(1-p)b_1\{E(B_1)+E(B_2)\}\\+r(1-r)b_1\{(1-p)E(B_1)+E(B_2\}\end{bmatrix}$$

$$D^{//}(1)=\left(\lambda b_1E(I)\right)^2\begin{bmatrix}2r(1-p)(1`+3r)E(B_1)E(B_2)\\+(1+r)(1-rp)E(B_1^2)\\+\left(1+r-2rp\right)E(B_2^2)\end{bmatrix}$$

$$+p(1-r)^2(\lambda b_2E(I))^2E(V^2)-p(1-r)b_2\lambda E(I/I-1)E(V)$$

$$+\{E(B_1)+E(B_2)\}\begin{bmatrix}2p(1-r)^2b_1b_2\left(\lambda E(I)\right)^2E(V)\\-b_1(1+r)\lambda E(I/I-1)\end{bmatrix}$$

$$+\{2E(B_1)+E(B_2)\}\begin{bmatrix}2pr(1-r)b_1b_2\left(\lambda E(I)\right)^2E(V)\\-b_1r(1-rp)\lambda E(I/I-1)\end{bmatrix}$$

where $E(B_1^2), E(B_2^2)$ are the second moments of first stage and second stage respectively, $E(V^2)$ is the second moment of vacation time and $E(I/I-1)$ is the second factorial moment of size of arriving batch of customers and $B_j^*(0)=1, j=1,2 \quad W^*(0)=1$

And Q has been obtained in (58)

Thus we can derive the mean queue size of the system by using the relation $L_s=\frac{d}{dz}P_S(z)$ at z=1.

Alternatively we can find, $L_S = L_Q + \rho$

Further we can obtain the average waiting time in the queue and system by the relations

$$W_Q = \frac{L_Q}{\lambda} \text{ and } W = \frac{L_S}{\lambda}$$

6. Particular cases:

1. $M^X/(G_1, G_2)/1$ Vacation Queue with balking and no re-service

Here we consider a queuing system with two stage heterogeneous service, vacation and balking but without re-service, thus taking r= 0 in our main results, we obtain

$$P_s(z) = \frac{(1-\rho)(1-z)\left[(1-p)+pW^*(b_2(\lambda-\lambda A(z)))\right]B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))}{\left[(1-p)+pW^*(b_2(\lambda-\lambda A(z)))\right]B_1^*(b_1(\lambda-\lambda A(z)))B_2^*(b_1(\lambda-\lambda A(z)))-z} \quad (63)$$

$$Q = 1-\lambda E(I)\left[\{b_1 E(B_1)+b_1 E(B_2)\}+pb_2 E(V)\right]$$

$$N^{/}(1) = Q\lambda E(I)\left[b_1\{E(B_1)+E(B_2)\}+pb_2 E(V)\right]$$

$$N^{//}(1) = Q\begin{bmatrix}\lambda E(I/I-1)\{b_1(E(B_1)+E(B_2))+pb_2 E(V)\} \\ -(\lambda E(I))^2\left[b_1^{\,2}\{E(B_1^2)+E(B_1^2)\}+pb_2^2 E(V^2)\right]\end{bmatrix}$$

$$D^{/}(1) = 1-\lambda E(I)\left[b_1\{E(B_1)+E(B_2)\}+pb_2 E(V)\right]$$

$$\begin{aligned} D^{//}(1) = {} & (\lambda E(I))^2\left[b_1^{\,2}\{E(B_1^2)+E(B_2^2)\}+pb_2^2 E(V^2)\right] \\ & +\lambda E(I/I-1)\left[b_1\{E(B_1)+E(B_2)\}-pb_2 E(V)\right] \\ & +2pb_1b_2(\lambda E(I))^2 E(V)\{E(B_1)+E(B_2)\} \end{aligned}$$

Thus equation (63) gives the steady state queue size of a $M^X/(G_1, G_2)/1$ with vacation and balking.

2. $M^X/(G_1, G_2)/1$ Vacation Queue with no re-service and no balking

Here we consider $r=0, b_1=b_2=1$, in the above relation (59), a queuing system offering no re-service and arriving customers does not balk , then our PGF reduces to

$$P_S(z)=\frac{(1-\rho)(1-z)\left[(1-p)+pW^*(\lambda-\lambda A(z))\right]B_1^*(\lambda-\lambda A(z))B_2^*(\lambda-\lambda A(z))}{\left[(1-p)+pW^*(\lambda-\lambda A(z))\right]B_1^*(\lambda-\lambda A(z))B_2^*(\lambda-\lambda A(z))-z} \quad (64)$$

$$Q=1-\lambda E(I)\left[E(B_1)+E(B_2)+pE(V)\right]$$

$$N^{/}(1)=Q\lambda E(I)\left[\{E(B_1)+E(B_2)\}+pE(V)\right]$$

$$N^{//}(1)=Q\begin{bmatrix}\lambda E(I/I-1)\{(E(B_1)+E(B_2))+pE(V)\}\\-(\lambda E(I))^2\left[E(B_1^2)+E(B_2^2)+pE(V^2)\right]\end{bmatrix}$$

$$D^{/}(1)=1-\lambda E(I)\left[E(B_1)+E(B_2)+pE(V)\right]$$

$$D^{//}(1)=(\lambda E(I))^2\left[\{E(B_1^2)+E(B_2^2)\}+pE(V^2)\right]$$
$$+\lambda E(I/I-1)\left[\{E(B_1)+E(B_2)\}-pE(V)\right]$$
$$+2p(\lambda E(I))^2E(V)\{E(B_1)+E(B_2)\}$$

The relation in (63) gives the steady state queue size of a $M^X/(G_1,G_2)/1/V_s$ without balking and re-service.

3. $M^X/(G_1,G_2)/1$ with no vacation, no balking and no re-service

We take $p=0, b_1=b_2=1, r=0$, i.e. we consider a queuing system where the server does not go for vacation, the arriving batch does not balk and the system does not offer re-service , then our PGF in (60) becomes

$$P_S(z)=\frac{(1-\rho)(1-z)B_1^*(\lambda-\lambda A(z))B_2^*(\lambda-\lambda A(z))}{B_1^*(\lambda-\lambda A(z))B_2^*(\lambda-\lambda A(z))-z} \quad (65)$$

$$Q=1-\lambda E(I)\{E(B_1)+E(B_2)\}$$

$$N^{/}(1)=Q\left[\lambda E(I)\{E(B_1)+E(B_2)\}\right]$$

$$N^{//}(1)=Q\left[\lambda E(I/I-1)\{E(B_1)+E(B_2)\}-(\lambda E(I))^2\{E(B_1^2)+E(B_2^2)\}\right]$$

$$D^{/}(1) = 1 - \lambda E(I)\left[E(B_1) + E(B_2)\right]$$

$$D^{//}(1) = \left(\lambda E(I)\right)^2\left[E(B_1^2) + E(B_2^2)\right] - \lambda E(I/I-1)\left[E(B_1) + E(B_2)\right]$$

The relation (65) gives the steady state queue size of a $M^X/(G_1, G_2)/1$ Queue.

4. $M^X/G_1/1/$ Queuing system with single server

In addition to the above case (3), if we assume that there is no second stage service, we achieve the result by obtained by Gaver [3] as

$$P_S(z) = \frac{(1-\rho)(1-z)B_1^*(\lambda - \lambda A(z)))}{B_1^*(\lambda - \lambda A(z))) - z} \tag{66}$$

$$Q = 1 - \lambda E(I)E(B_1)$$

$$N^{/}(1) = Q\lambda E(I)E(B_1)$$

$$N^{//}(1) = Q\left[\lambda E(I/I-1)E(B_1) - \left(\lambda E(I)\right)^2 E(B_1^2)\right]$$

$$D^{/}(1) = 1 - \lambda E(I)E(B_1)$$

$$D^{//}(1) = \left(\lambda E(I)\right)^2 E(B_1^2) - \lambda E(I/I-1)E(B_1)$$

The above relation(66) gives the steady state queue size of a $M^X/G/1$ queue.

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