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A Note on Nonwandering Set of a Nonautonomous Discrete Dynamical System

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Abstract

In this paper, we study nonwandering set and recurrent set for the non-autonomous discrete dynamical system given by a sequence $\{f_n\}_{n=0}^{\infty}$ of homeomorphisms on a compact metric space.

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1 Introduction

Nonautonomous discrete dynamical systems induced by sequence $\{f_n\}_{n=0}^{\infty}$ of continuous self-maps on a compact metric space have been studied in detail e.g. [2, 4]. In various problems, including applied ones, we often work with a sequence of maps instead of a single map. For example, if we take computational experiment or we study random dynamical system then the map used at a given time is chosen from a given set. In the study of dynamics of a map f from a compact metric space X to itself, central role is played by the various recursive properties of points of X. One of important such properties is nonwanderingness. Nonwandering points play important role in study of autonomous and nonautonomous discrete dynamical systems. For study of nonwandering points for autonomous discrete dynamical system, one can refer [1]. In [3], authors have studied nonwandering set from the view-point of topological entropy. In [2, 7], authors have studied structures of nonwandering sets in nonautonomous discrete systems.

In the next section we define and study nonwandering set, α-limit set, ω-limit set and recurrent set for nonautonomous discrete dynamical systems. Our definition of nonwandering point is different form those given in [3] and [7]. For the time varying dynamical systems the terms orbit, periodicity and expansiveness are defined in [4]. We have studied expansiveness, shadowing and topological stability for a nonautonomous discrete system induced by a sequence of continuous maps on a compact metric space in [5] and for a nonautonomous discrete system induced by a sequence of homeomorphisms on a compact metric space in [6].

Throughout this article we consider (X,d) to be a metric space and $f_n : X \to X$, $n = 0,1,2,...$ to be a sequence of homeomorphisms, where f_0 is the identity map on X. We call (X,F) time varying homeomorphism or an invertible nonautonomous discrete dynamical system. We denote

$$F_n = \begin{cases} f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0, & for \ \ n \geq 0 \\ f_{-n}^{-1} \circ f_{-(n-1)}^{-1} \circ \cdots \circ f_1^{-1} \circ f_0^{-1} & for \ \ n \leq -1. \end{cases}$$

Now for time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$, it's inverse action is the time varying homeomorphism $F^{-1} = \{f_n^{-1}\}_{n=0}^{\infty}$. We define,

$$F_{[i,j]} = \begin{cases} f_j \circ f_{j-1} \circ \cdots \circ f_{i+1} \circ f_i, & for \ \ i \leq j \\ identity \ \ map & for \ \ i > j. \end{cases}$$

Thus

$$F_{[i,j]}^{-1} = \begin{cases} f_j^{-1} \circ f_{j-1}^{-1} \circ \cdots \circ f_{i+1}^{-1} \circ f_i^{-1} & for \ \ i \leq j \\ identity \ \ map & for \ \ i > j. \end{cases}$$

Definition 1.1 *[6] Let (X,d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphisms, $n = 0,1,2,....$ For a point $x_0 \in X$, define a sequence as follows :*

$$x_n = \begin{cases} f_n(x_{n-1}), & if \ \ n \geq 1; \\ f_{-n}^{-1}(x_{n+1}), & if \ \ n < 0. \end{cases}$$

Then the sequence $O(x_0) = \{x_n\}_{n=-\infty}^{\infty}$ is said to be the orbit of x_0 under time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$. Note that $O(x_0) = \{F_n(x_0) \colon n \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of all integeres.

2 Nonwandering Set

Definition 2.1 *Let (X,d) be a metric space and $f_k : X \to X$ be a sequence of homeomorphisms, $k = 0,1,2,....$ A point $x \in X$ is said to be* **a nonwandering point** *for time varying homeomorphism $F = \{f_k\}_{k=0}^{\infty}$ if for any neighborhood U of x and for any $n \geq 0$ there exists $m \geq n$ and $r \geq 0$ such that $F_{[m,m+r]}(U) \cap U \neq \phi$ or $F^{-1}_{[m,m+r]}(U) \cap U \neq \phi$. The set of all nonwandering points is denoted by $\Omega(F)$.*

Theorem 2.1 *Let (X,d) be a compact metric space and $F = \{f_n\}_{=0}^{\infty}$ be the time varying homeomorphism on X, then $\Omega(F)$ is a nonempty set.*

Proof : Suppose $\Omega(F)$ is empty. Then for any $x \in X$, $x \notin \Omega(F)$. Therefore there exist a neighborhood U_x of x and an integer $n_x > 0$ such that for any $y \in U_x$, $F_{[m_x,m_x+r_x]}(y) \notin U_x$ and $F^{-1}_{[m_x,m_x+r_x]}(y) \notin U_x$ for every $m_x \geq n_x$ and $r_x \geq 0$. Note that $\{U_x : x \in X\}$ forms an open cover of X. Since X is compact, there exist $x_0, x_1, ..., x_m$ such that $\cup_{k=0}^{m} U_{x_k} = X$. Let $N = \max\{n_{x_i} : i = 0,1,\cdots,m\}$. Now for any $z \in U_{x_0}$, $F_{[N,N+1]}(z) \notin U_{x_0}$ (for $m_{x_0} = N \geq n_{x_0}$ and $r_{x_0} = 1$), say $F_{[N,N+1]}(z) \in U_{x_1}$. Then

$$F_{[N,N+2]}(z) = F_{[N+1,N+2]}(F_{[N,N+1]}(z)) \notin U_{x_0} \cup U_{x_1}$$

(for $m_{x_0} = N \geq n_0$, $r_{x_0} = 2$ and for $m_{x_1} = N+1 > n_{x_1}$, $r_{x_1} = 1$). Continuing in this way, we have $F_{[N,N+m+1]}(z) \notin \cup_{i=0}^{m} U_{x_i} = X$ which gives a contradiction. Hence $\Omega(F)$ must be nonempty.

Theorem 2.2 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X,d). Then $\Omega(F)$ is closed set.*

Proof Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in $\Omega(F)$ converges to $x \in X$. Let $\delta > 0$ and $s \in \mathbb{N}$. Since $x_n \to x$, $\exists\ l \geq 0$ such that $x_l \in U_\delta(x) = \{y \in X : d(x,y) < \delta\}$. Since $U_\delta(x)$ is open, there exists $\epsilon > 0$ such that $U_\epsilon(x_l) \subset U_\delta(x)$. Since x_l is a nonwandering point, there exist $m \geq s$ and $r \geq 0$ such that $F_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi$ or $F^{-1}_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi$. Since $U_\epsilon(x_l) \subset U_\delta(x)$, we have $F_{[m,m+r]}(U_\epsilon(x_l)) \subset F_{[m,m+r]}(U_\delta(x))$ and $F^{-1}_{[m,m+r]}(U_\epsilon(x_l)) \subset F^{-1}_{[m,m+r]}(U_\delta(x))$. Thus

$$F_{[m,m+r]}(U_\delta(x)) \cap U_\delta(x) \supset F_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi.$$

or

$$F^{-1}_{[m,m+r]}(U_\delta(x)) \cap U_\delta(x) \supset F^{-1}_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi.$$

Thus $x \in \Omega(F)$ and hence $\Omega(F)$ is closed.

Definition 2.2 *Let (X,d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphisms, $n = 0,1,2,....$ A point $x_0 \in X$ is said to be periodic point of time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ if orbit of x_0 is periodic, i.e. there exists an integer $k > 0$ such that $F_{ik+j}(x_0) = F_j(x_0)$, for every $i \in \mathbb{Z}$ and $0 \leq j < k$. The set of all periodic points of F is denoted by $Per(F)$.*

Definition 2.3 *Let (X,d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphisms, $n = 0, 1, 2, \ldots$. A point $x \in X$ is said to be fixed point of time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ if $f_n(x) = x$, for all $n \geq 0$.*

Theorem 2.3 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X,d). Then $Per(F) \subset \Omega(F)$.*

Poof Let $x \in Per(F)$ then there exists $k > 0$ such that $F_{ik+j}(x) = F_j(x)$ for every $i \geq 0$ and $0 \leq j < k$. Now for any $n > 0$ there exists $i > 0$ such that $ik \geq n$, Now $F_{ik}(x) = x$ and $F_{(i+1)k}(x) = x$. i.e. $F_{[ik+1,(i+1)k]}(x) = F_{[ik+1,(i+1)k]}(F_{ik}(x)) = F_{(i+1)k}(x) = x$. Thus for any neighborhood U of x and any $n > 0$ there exist $m = ik + 1 > n$ and $r = k - 1 \geq 0$ such that $F_{[ik+1,ik+1+(k-1)]}(U) \cap U \neq \phi$. Hence $x \in \Omega(F)$.

Definition 2.4 *Let (X,d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphisms, $n = 0, 1, 2, \ldots$. The time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ is said to be* **topologically strongly transitive** *if for any for any two nonempty open sets $U, V \subset X$,and any $n \geq 0$ there is $m \geq n$ and $r \geq 0$ such that*

$$F_{[m,m+r]}(U) \cap V \neq \phi \ \ or \ \ F^{-1}_{[m,m+r]}(U) \cap V \neq \phi$$

Theorem 2.4 *Let (X,d) be a metric space and $F = \{f_n\}_{=0}^{\infty}$ be a time varying homeomorphism on X. If F is topologically strongly transitive then $\Omega(F) = X$.*

Proof : Let $x \in X$, U_x be a neighborhood of x and $n > 0$ be given. Since F is topologically strongly transitive, for any two nonempty open sets $U, V \subset X$, there exist $m \geq n$ and $r \geq 0$ such that $F_{[m,m+r]}(U) \cap V \neq \phi$ or $F^{-1}_{[m,m+r]}(U) \cap V \neq \phi$. In particular for $V = U = U_x$ we have $F_{[m,m+r]}(U_x) \cap U_x \neq \phi$. Thus $x \in \Omega(F)$. Hence $\Omega(F) = X$.

Definition 2.5 *Let (X,d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. By α-* **limit set of a point** *$x \in X$, we mean the set*

$$\alpha(x) = \{y \in X | \lim_{k \to \infty} d(F_{n_k}(x), y) = 0\},$$

where $\{n_k\}$ is some strictly decreasing sequence of negative integers.

Definition 2.6 *Let (X,d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. By ω-* **limit set of a point** *$x \in X$, we mean the set*

$$\omega(x) = \{y \in X | \lim_{k \to \infty} d(F_{n_k}(x), y) = 0\},$$

where $\{n_k\}$ is some strictly increasing sequence of positive integers.

Definition 2.7 *Let (X, d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. A point $x \in X$ is said to be* **recurrent** *if $x \in \alpha(x) \cap \omega(x)$. We denote the set of all recurrent points of F by $\mathcal{R}(F)$ and the closure of it by $C(F)$.*

Remark 2.1 *It is clear from the definition of $Per(F)$ and $\mathcal{R}(F)$ that $Per(F) \subset \mathcal{R}(F)$.*

Theorem 2.5 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. Then for any $x \in X$, $\omega(x) \subset \Omega(F)$ and $\alpha(x) \subset \Omega(F)$.*

Proof : Let $x \in X$ and $y \in \omega(x)$ be given. Let $n > 0$ and U be a neighborhood of y. Then there exist an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $F_{n_k}(x) \to y$. Then there exist $n_j > n_i > n$ such that $F_{n_i}(x) \in U$ and $F_{n_j}(x) \in U$,i.e. $F_{[n_i+1,n_j]}(F_{n_i(x)}) \in U$. Thus $F_{[n_i+1,n_i+1+(n_j-n_i-1)]}(U) \cap U \neq \phi$, where $n_i+1 \geq n$ and $(n_j-n_i-1) \geq 0$. Thus $y \in \Omega(F)$. Therefore $\omega(x) \subset \Omega(F)$, for any $x \in X$.
Similarly let $x \in X$ and $y \in \alpha(x)$ be given. Let $n > 0$ and U be a neighborhood of y. Then there exist a decreasing sequence $\{n_k\}_{k=0}^{\infty}$ of negative integers such that $F_{n_k}(x) \to y$. Then there exist $n_j < n_i < -n$ such that $F_{n_i}(x) \in U$ and $F_{n_j}(x) \in U$, i.e. $F^{-1}_{[-n_i+1,-n_j]}(F_{n_i(x)}) \in U$. Thus $F^{-1}_{[-n_i+1,-n_i+1+(n_i-n_j-1)]}(U) \cap U \neq \phi$, where $-n_i + 1 \geq n$ and $(n_i - n_j - 1) \geq 0$. Thus $y \in \Omega(F)$. Therefore $\alpha(x) \subset \Omega(F)$, for any $x \in X$.

Remark 2.2 *From the definition of $\mathcal{R}(F)$ for any $x \in \mathcal{R}(F)$, $x \in \alpha(x) \cap \omega(x)$ and form above theorem $\alpha(x) \cap \omega(x) \subset \Omega(F)$. Thus $\mathcal{R}(F) \subset \Omega(F)$. For a compact metric space $\Omega(F)$ is closed, then we have $C(F) \subset \Omega(F)$.*

Example 2.1 *We construct a time varying homeomorphism for which every point is recurrent and hence nonwandering but not all points are periodic. Hence set of all periodic points is a proper subset of set of all nonwandering points.*

Let $X = [0, 1]$ and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism such that

$$f_n(x) = \begin{cases} x & for \;\; n = i^2, \;\; i = 1, 2, 3, \ldots; \\ x^{1/(k+1)} & for \;\; n = i^2 + k, \;\; k = 1, 2, \ldots \;\; where \;\; i = k, k+1, k+2, \ldots; \\ x^{k+1} & for \;\; n = (i+1)^2 - k, \;\; k = 1, 2, \ldots \;\; where \;\; i = k, k+1, k+2, \ldots. \end{cases}$$

Thus we have sequence $\{x, \sqrt{x}, x^2, x, \sqrt{x}, \sqrt[3]{x}, x^3, x^2, x, \ldots\}$ of homeomorphisms. Clearly each $x \in [0, 1]$ is a recurrent point as it infinitely many times re-appears in its orbit and period of appearance is getting large and large, except for 0 and 1 which are fixed points. So no point except 0 and 1 is periodic. Thus

$Per(F)$ is proper subset of $\mathcal{R}(F)$.

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