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The Exponential Function as a Limit

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Abstract

In this paper we define the exponential function of base e and we establish its basic properties.We also define the logarithmic function of base e and we prove its continuity.

Keywords: number e, limit of sequence of functions, exponential function, logarithmic function

1 Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers and let $\mathbb{R}$ be the set of real numbers. Suppose that $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of functions defined on $E \subseteq \mathbb{R}$. We say that this sequence converges to the function $f(x)$ on E if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for any} \quad x \in E.$$

This means that

$$\forall\, \varepsilon > 0\, \forall\, x \in E\, \exists\, N = N(\varepsilon, x) \in \mathbb{N}\; \forall\, n \in \mathbb{N} : n > N \Rightarrow |\, f_n(x) - f(x)\,| < \varepsilon \tag{1.1}$$

In this case we write $f_n(x) \overset{E}{\to} f(x)$ $(n \to \infty)$.
Suppose that $x \in \mathbb{R}$. Let us consider the numbers $m_0 = m_0(x)$ and $n_0 = n_0(x)$ defined as follows :

$$m_0 = m_0(x) = \{k \in \mathbb{N} \mid \; k > x\} \;\text{ and }\; n_0 = n_0(x) = \{k \in \mathbb{N} \mid \; k > -x\} \tag{1.2}$$

Thus, $m_0 = 1$ and $n_0 = [-x] + 1$ if $x \leq 0$ and $m_0 = [x] + 1$ and $n_0 = 1$ if $x \geq 0$ ($[x]$ is the integer part of x) . It is clear that $1 + \dfrac{x}{n} > 0$ for any $n \geq n_0$

and $1 - \frac{x}{n} > 0$ for any $n \geq m_0$. We define the two sequences $\{ f_n(x) \}_{n=1}^{\infty}$ and $\{ g_n(x) \}_{n=1}^{\infty}$ as follows :

$$f_n(x) = 0 \text{ if } n < n_0 \text{ and } f_n(x) = \left(1 + \frac{x}{n}\right)^n \text{ if } n \geq n_0. \tag{1.3}$$

Moreover,

$$g_n(x) = 0 \text{ if } n < m_0 \text{ and } g_n(x) = \left(1 - \frac{x}{n}\right)^{-n} \text{ if } n \geq m_0. \tag{1.4}$$

Lemma 1. Let $x \in \mathbb{R}$ and consider the sequences defined by (1.3) and (1.4).
a. The sequence $\{ f_n(x) \}_{n=1}^{\infty}$ is increasing for $n \geq n_0$, that is, $f_n(x) \leq f_{n+1}(x)$ for any $n \geq n_0$. In particular it is increasing for $x \geq 0$ since $n_0 = 1$.
b. The sequence $\{ g_n(x) \}_{n=1}^{\infty}$ is decreasing for $n \geq m_0$, that is, $g_n(x) \geq g_{n+1}(x)$ for any $n \geq m_0$. In particular it is increasing for $x \leq 0$ since $m_0 = 1$.
c. $0 \leq g_n(x) - f_n(x) \leq \frac{x^2}{n} g_{k_0}(x)$ for any $n \geq k_0 = \max(m_0, n_0)$.
d. There exist the limits $\lim_{n\to\infty} f_n(x) = \sup\{ f_n(x) \,|\, n \in \mathbb{N} \}$ and $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = L$. Moreover, $f_{n_0}(x) \leq L \leq g_{m_0}(x)$
e. If $|h| < 1$ then

$$1 + h \leq \left(1 + \frac{h}{n}\right)^n \leq \left(1 - \frac{h}{n}\right)^{-n} \leq (1 - h)^{-1} \text{ for all } n \geq 1 \tag{1.5}$$

Proof.
a. Let $n \geq n_0$. From the AGM inequality

$$\frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} \geq \sqrt[n+1]{a_1 a_2 \cdots a_{n+1}} \ (a_i > 0, \ i = 1, 2, ..., n+1) \tag{1.6}$$

with

$$a_1 = 1, \quad a_2 = a_3 = \cdots = a_{n+1} = 1 + \frac{x}{n} > 0$$

we obtain

$$1 + \frac{x}{n+1} = \frac{1 + n(1 + \frac{x}{n})}{n+1} \geq \sqrt[n+1]{\left(1 + \frac{x}{n}\right)^n}$$

and then

$$f_{n+1}(x) = \left(1 + \frac{x}{n+1}\right)^{n+1} \geq \left(1 + \frac{x}{n}\right)^n = f_n(x).$$

This inequality is strict unless $x = 0$.

b. Let $n \geq m_0$. From the AGM inequality (1.6) with

$$a_1 = 1, \quad a_2 = a_3 = \cdots = a_{n+1} = 1 - \frac{x}{n} > 0.$$

it follows that

$$1 - \frac{x}{n+1} = \frac{1 + n(1 - \frac{x}{n})}{n+1} \geq \sqrt[n+1]{\left(1 - \frac{x}{n}\right)^n},$$

and then

$$\left(1 - \frac{x}{n+1}\right)^{n+1} \geq \left(1 - \frac{x}{n}\right)^n > 0,$$

which implies

$$g_{n+1}(x) = \left(1 - \frac{x}{n+1}\right)^{-(n+1)} \leq \left(1 - \frac{x}{n}\right)^{-n} = g_n(x).$$

This inequality is strict unless $x = 0$.
c. Let $n \geq k_0 = \max(m_0, n_0)$. We have

$$g_n(x) - f_n(x) = g_n(x) \left(1 - \frac{f_n(x)}{g_n(x)}\right) = g_n(x) \left(1 - q^n\right), \tag{1.7}$$

where $q = 1 - \frac{x^2}{n^2}$. Observe that $n \geq k_0 > |x|$ from where $0 < q \leq 1$ and then $q^n \leq 1$ and $1 - q^n \geq 0$. It is clear from (1.7) that $g_n(x) - f_n(x) \geq 0$ for $n \geq k_0$. On the other hand, by virtue of (1.7),

$$\begin{aligned} 0 \leq g_n(x) - f_n(x) &= g_n(x)(1-q)\left(1 + q + \cdots + q^{n-1}\right) \\ &\leq g_{k_0}(x) \cdot \tfrac{x^2}{n^2}(1 + 1 + \cdots + 1) \\ &= g_{k_0}(x) \cdot \tfrac{x^2}{n^2} \cdot n = \tfrac{x^2}{n} g_{k_0}(x). \end{aligned}$$

Thus,

$$0 \leq g_n(x) - f_n(x) \leq \frac{x^2}{n} g_{k_0}(x) \text{ for } n \geq k_0. \tag{1.8}$$

From the last inequality we see that given $\varepsilon > 0$ if we choose a natural number N subject to $N \geq k_0$ and $N > x^2 g_{k_0}(x)/\varepsilon$ then

$$|g_n(x) - f_n(x)| = g_n(x) - f_n(x) < \varepsilon \quad \text{for all} \quad n > N.$$

We have proved that

$$\lim_{n \to \infty} \left(g_n(x) - f_n(x)\right) = 0. \tag{1.9}$$

d. Let $k_0 = \max(m_0, n_0) \geq m_0$. By virtue of **b** and **c**,

$$g_n(x) \leq g_{k_0}(x) \text{ and } f_n(x) \leq g_n(x) \leq g_{k_0}(x) \text{ for all } n \geq k_0,$$

which proves that the sequence $\{ f_n(x) \}_{n=1}^{\infty}$ is bounded from above for each $x \in \mathbb{R}$ and then $\lim_{n\to\infty} f_n(x) = L$, where

$$L = \sup\{ f_n(x) \,|\, n \in \mathbb{N} \} = \sup\{ f_n(x) \,|\, n \geq n_0 \}.$$

On the other hand, from (1.9),

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} ((g_n(x) - f_n(x)) + f_n(x)) = L.$$

e. Observe that $m_0 = n_0 = 1$ since $|h| < 1$. From **a**, **b** and **c** we obtain

$$1 + h = f_1(h) \leq f_n(h) \leq g_n(h) \leq g_1(h) = (1-h)^{-1} \text{ for any } n \geq k_0 = 1.$$

2 The exponential function and its properties

In previous section we established the existence of the limits

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^{-n}$$

for each $x \in \mathbb{R}$. This allows us to define a function $\exp : \mathbb{R} \to (0, \infty)$ as follows :

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^{-n}, \; x \in \mathbb{R}. \tag{2.1}$$

Is obvious that $\exp(0) = 1$. The value $\exp(1)$ is special and it is denoted by e :

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828182846$$

We well call function defined by (2.1) the exponential function of base e. This function is also denoted by e^x.

2.1 Properties of the exponential function

In this section we establish the main properties of the exponential function starting from (2.1).

Property 1. Let $x \in \mathbb{R}$.

i. If $x > -1$ then $\exp(x) > 1 + x$. In particular, $\exp(x) > 1$ for $x > 0$.

ii. If $x < 1$ then $\exp(x) \le \dfrac{1}{1-x}$. In particular, $\exp(x) < 1$ if $x < 0$.

Proof .

i. Since $x > -1$ we have $n_0 = [-x] + 1 = 1$. By virtue of lemma 1, parts **a** and **d**,

$$\exp(x) \ge \left(1 + \frac{x}{2}\right)^2 > \left(1 + \frac{x}{1}\right)^1 = 1 + x.$$

ii. If $x < 1$ then $m_0 = [x] + 1 = 1$. In view of lemma 1 parts **b**, **c** and **d**, $f_n(x) \le g_n(x) \le g_1(x)$ for all $n \ge k_0 = \max(m_0, n_0)$ and letting $n \to \infty$ we obtain

$$\exp(x) = \lim_{n\to\infty} f_n(x) \le g_1(x) = \left(1 - \frac{x}{1}\right)^{-1} = \frac{1}{1-x}.$$

Property 2. (Multiplicative property)

$$\exp(x + y) = \exp(x)\exp(y) = \exp(y)\exp(x) \quad \text{for any} \quad x, y \in \mathbb{R}. \tag{2.2}$$

In particular,

$$\exp(-x) = (\exp(x))^{-1} = \frac{1}{\exp(x)} \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.3}$$

Proof . Let us consider the sequences

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n, \; f_n(y) = \left(1 + \frac{y}{n}\right)^n \text{ and } \; f_n(x + y) = \left(1 + \frac{x + y}{n}\right)^n,$$

where $n \ge k_0 > |x| + |y|$. By lemma 1, part **d**,

$$\lim_{n\to\infty} f_n(x) = \exp(x), \quad \lim_{n\to\infty} f_n(y) = \exp(y) \text{ and } \lim_{n\to\infty} f_n(x + y) = \exp(x + y).$$

Since $h(n) \overset{\text{def}}{=} \dfrac{xy}{n + x + y} \to 0$ ($n \to \infty$),we may choose N large enough so that $|h(n)| < 1$ for $n \ge N$ .We obtain

$$\frac{f_n(x)f_n(y)}{f_n(x + y)} = \left(1 + \frac{xy}{n(n + x + y)}\right)^n = \left(1 + \frac{h(n)}{n}\right)^n \text{ for } n \ge N. \tag{2.4}$$

In view of lemma 1, part **e**, from (2.4) it is clear that

$$1 + h(n) \le \frac{f_n(x)f_n(y)}{f_n(x + y)} \le (1 - h(n))^{-1} \tag{2.5}$$

Taking into account that $\lim_{n\to\infty}(1+h(n)) = \lim_{n\to\infty}(1-h(n))^{-1} = 1$ from (2.5) we obtain $\lim_{n\to\infty}\frac{f_n(x)f_n(y)}{f_n(x+y)} = 1$, from where

$$\frac{\exp(x)\exp(y)}{\exp(x+y)} = \frac{\lim_{n\to\infty} f_n(x)\lim_{n\to\infty} f_n(y)}{\lim_{n\to\infty} f_n(x+y)} = \frac{\lim_{n\to\infty} f_n(x)f_n(y)}{\lim_{n\to\infty} f_n(x+y)} = 1.$$

We have proved that $\exp(x)\exp(y) = \exp(x+y)$.
Property 3. Given $t,\, x \in \mathbb{R}$, if $t < x$, then $\exp(t) < \exp(x)$, that is, the exponential function is strictly increasing on $\mathbb{R}$.
Proof . If $x > t$ then $x - t > 0$ and making use of Property 1, $\exp(x-t) > 1$. We have

$$\exp(x) = \exp((x-t)+t) = \exp(x-t)\exp(t) > 1\cdot\exp(t) = \exp(t).$$

Property 4. If $x > 0$ then $0 < \exp(x) - 1 \le x\exp(x)$.
Proof . Let $n \in \mathbb{N}$. We have

$$\begin{aligned} 0 < \left(1+\frac{x}{n}\right)^n - 1 &= \left(1+\frac{x}{n}-1\right)\left(\left(1+\frac{x}{n}\right)^{n-1} + \left(1+\frac{x}{n}\right)^{n-2} + \cdots + 1\right) \\ &< \frac{x}{n}\left(\left(1+\frac{x}{n}\right)^n + \left(1+\frac{x}{n}\right)^n + \cdots + \left(1+\frac{x}{n}\right)^n\right) \\ &= \frac{x}{n}\cdot n\left(1+\frac{x}{n}\right)^n = x\left(1+\frac{x}{n}\right)^n < x\exp(x). \end{aligned}$$

Thus,

$$0 < \left(1+\frac{x}{n}\right)^n - 1 < x\exp(x) \quad \text{for any} \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ in the last inequality gives

$$0 < \exp(x) - 1 \le x\exp(x). \tag{2.6}$$

Property 5. The exponential function is continuous on $\mathbb{R}$, i.e, for a given real number a and any $\varepsilon > 0$ we may find $\delta = \delta(\varepsilon, a) > 0$ such that if $|x-a| < \delta$ then $|\exp(x) - \exp(a)| < \varepsilon$.
Proof . Let us first show that

$$|\exp(t) - 1| \le 3|t| \quad \text{for} \quad |t| < 1. \tag{2.7}$$

Indeed, this inequality is obvious if $t = 0$. Let $t \ne 0$. If $0 < t < 1$ then $\exp(t) < \exp(1) = e < 3$. Consequently, in view of Property 4, $0 < \exp(t) - 1 < 3t$.

Now, let $-1 < t < 0$. From one hand, by Property 1, $\exp(t) < 1$.On the other hand, $0 < -t < 1$ and then $0 < \exp(-t) - 1 < 3(-t) = 3|t|$, from where

$$|\exp(t) - 1| = |\exp(t)(1 - \exp(-t))| = \exp(t)(\exp(-t) - 1) < 3\exp(t)|t| < 3|t|.$$

We have established (2.7). Let $a \in \mathbb{R}$ and consider values of x subject to $|x - a| < 1$. Setting $t = x - a$ in (2.7) we obtain $|\exp(x - a) - 1| < 3|x - a|$. Multiplying this inequality by $\exp(a)$ and making use of Property 2 we obtain

$$|\exp(x) - \exp(a)| < 3\exp(a)|x - a| \text{ for any } x \in \mathbb{R} \text{ such that } |x - a| < 1. \tag{2.8}$$

From condition (2.8) it is clear that choosing δ such that

$$0 < \delta < \min\left(\frac{1}{2}, \frac{\varepsilon}{3\exp(a)}\right),$$

then $|\exp(x) - \exp(a)|$ for all $x \in \mathbb{R}$ such that $|x - a| < \delta$. This means that $\lim_{x \to a} \exp(x) = \exp(a)$.

3 The logarithmic function

In view of Property 5, the exponential function is strictly increasing on $\mathbb{R}$. In view of Property 1, part i, $\exp(x) > 1 + x > x$ for $x \geq 0$. On the other hand, if $x < 0$ then (2.2) gives $\exp(x)\exp(-x) = \exp(x - x) = \exp(0) = 1$ and then $\exp(x) > 0$. This says that the exponential function $\exp : \mathbb{R} \to (0, \infty)$ is one to one and it admits inverse We will denote it by log and we will call it logarithmic function of base $e : \log : (0, \infty) \to \mathbb{R}$.

Let $y \in (0, \infty)$. There exists $x \in \mathbb{R}$, uniquely defined, such that $\exp(x) = y$. Indeed, choose $b > 0$ subject to $b > y - 1$. By Property 1, part **i**, $\exp(b) > 1 + b > y$.

On the other hand, let a be any negative number such that $a < 1 - 1/y$. By Property 1, part **ii**,

$$\exp(a) \leq \frac{1}{1 - a} < y.$$

We have proved that for any $y \in (0, \infty)$ we may find two real numbers a and b such that $a < b$ and $\exp(a) < y < \exp(b)$. Let us consider the function $\exp(x)$ on the interval $[a, b]$. Since this function is continuous on $[a, b]$ (Property 1), it takes all values between $\exp(a)$ and $\exp(b)$. This allows us to choose x on $[a, b]$ for which $y = \exp(x)$. This number x is unique since the exponential function is one to one. We have proved that function $\exp : \mathbb{R} \to (0, \infty)$ is one to one and onto. Thus, $\log(y) = x \Leftrightarrow y = \exp(x)$. It is clear that

$$\exp(\log(y)) = y \ \forall\, y > 0 \ \text{ and } \ \log(\exp(x)) = x \ \forall\, x \in \mathbb{R}.$$

Theorem. The logarithmic function $\log : (0, \infty) \to \mathbb{R}$ is continuous on $(0, \infty)$.
Proof : It is easy to see that given $b > 0$ and $\varepsilon > 0$ if $|y - b| < \delta = \min\left(b(1 - \exp(-\varepsilon))\, ,\, b(\exp(\varepsilon) - 1)\right)$ then $|\log(y) - \log(b)| < \varepsilon$.This means that $\lim\limits_{y \to b} \log(y) = \log(b)$ for any $b > 0$.

4 Conlusions

We defined two of the most important functions in mathematics: the exponential and logarithmic functions. This allows to define the exponential function of base a as s $a^x = \exp(x \log(a)), x \in \mathbb{R}, a > 0$ and $a \neq 1$. From this we may establish the laws of exponents :

$$a.\, a^x a^y = a^{x+y}; \quad b.\, \frac{a^x}{a^y} = a^{x-y}; \quad c.\, a^x b^x = (ab)^x; \quad d.\, \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x; \ e.\, (a^x)^y = a^{xy}.$$

Finally, we may define the logarithmic function as a limit as follows

$$\log(x) = \lim_{n \to \infty} n(\sqrt[n]{x} - 1) = \lim_{n \to \infty} n(1 - x^{-1/n}) \text{ for } x > 0.$$

Starting from this definition we may define the exponential function as the inverse of logarithmic function.

References

[1] Hardy G. H., *A Course of Pure mathematics*, Ninth Edition, University of Cambridge (1945).

[2]Mitrinovic D.S., *Elementary Inequalities*, Belgrade-Yugoslavia, P. Noordhoff Ltd - Groningen - The Netherlands (1964)

[3] Polya G. & Szego G., *Aufgaben und lehrsatze aus der analysis*, Springer Verlag, Berlin (1964).

[4] Rudin W., *Principles of Mathematical Analysis*, Third Edition, McGraw Hill (1976).

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