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New Exact Solutions of Stochastic KdV Equation

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Abstract

In this paper, the multi-wave method is used to find new exact solitary solutions of nonlinear stochastic KdV equation. By using this approach and with the help of Mathematica we obtain a new type of multi-wave solution, periodic one and two-solitary-wave solution for the stochastic KdV equation are researched. The results presented in this paper improve the previous results.

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1 Introduction

Recently, random waves are very important subjects of stochastic partial differential equations. Firstly, Wadati investigated stochastic partial differential KdV equation, and gave the diffusion of soliton of KdV equation with the Gaussian noise in [1]. Also stochastic equations have been studied by many authors [2-7], and so on. Consequently, construction of traveling wave solutions of stochastic differential equations plays an important role in nonlinear science. Various methods for seeking traveling wave solutions to nonlinear partial differential equations are proposed such as homogeneous balance method, exp-function method, $(\frac{G'}{G})$-expansion method, extended three-soliton method, trial equation method [8-17], and so on. In [18], Y. Shi et al. proposed and developed multi-wave method to construct a series of exact solutions of nonlinear partial differential equations including periodic solitary wave solutions, bright soliton wave solutions, M-type wave solutions, etc. In this study, we apply this method to find exact solitary solutions for stochastic KdV equation.

The rest of this paper is organized as follows. In Section 2, we obtain the bilinear form of stochastic KdV equation. In Section 3, we give the multi-wave method to solve nonlinear stochastic nonlinear partial differential equations. Also, we apply this improved algorithm to the stochastic KdV equation, and obtain new exact solutions to this equation. In Section 4, some conclusions are given.

2 Bilinear form of stochastic KdV equation

In this paper we will give some new exact solutions of stochastic KdV equation with time dependent noise as the following form [1]:

$$U_T + 6UU_X + U_{XXX} = \zeta(T) \tag{1}$$

One simply applies the Galilean transformation

$$U(X,T) = u(x,t) + w(t), \quad x = X + m(T), \quad t = T, \tag{2}$$

$$m(T) = -6\int_0^T W(T')dT', \qquad W(T) = \int_0^T \zeta(T')dT', \tag{3}$$

to transform the stochastic KdV into KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{4}$$

Let's suppose

$$u(x,t) = w_x(x,t) = (\ln f)_{xx}, \quad w = (\ln f)_x. \tag{5}$$

Substituting Eq. (5) into Eq. (4), we can have

$$w_{xt} + 6w_x w_{xx} + w_{xxxx} = 0. \tag{6}$$

Integrating Eq. (6) with respect to x once yields

$$w_t + 3w_x^2 + w_{xxx} = 0, \tag{7}$$

where the integration constant is zero. Eq. (7) can be expressed in terms of the bilinear operators [18-20]

$$D_x(D_t + D_x^3)f.f = 0, \tag{8}$$

with the customary definition of the Hirota's bilinear operators

$$D_t^n D_x^m a.b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m a(x,t)b(x',t')|_{x'=x,\ t'=t}. \tag{9}$$

3 Multi-wave method for the stochastic KdV equation

Now, we assume that the test function of multi-wave method can be expressed in the following form [18]:

$$f(x,t) = a_1\cos(\xi_1) + a_2\sin(\xi_2) + a_3\cosh(\xi_3) + \exp(-\xi_4) + a_4\exp(\xi_4), \tag{10}$$

where $\xi_i = p_i x + r_i t$, $i = 1, 2, 3, 4$. Substituting Eq. (10) into Eq. (8), we can obtain the form of general solution of Eq. (4):

$$u = -\frac{(-a_1p_1\sin(\xi_1) + a_2p_2\cos(\xi_2) + a_3p_3\sinh(\xi_3) - p_4e^{-\xi_4} + a_4p_4e^{\xi_4})^2}{(e^{-\xi_4} + a_1\cos(\xi_1) + a_2\sin(\xi_2) + a_3\cosh(\xi_3) + a_4e^{\xi_4})^2} + \frac{-a_1p_1^2\cos(\xi_1) - a_2p_2^2\sin(\xi_2) + a_3p_3^2\cosh(\xi_3) + p_4^2e^{-\xi_4} + a_4p_4^2e^{\xi_4}}{e^{-\xi_4} + a_1\cos(\xi_1) + a_2\sin(\xi_2) + a_3\cosh(\xi_3) + a_4e^{\xi_4}}. \tag{11}$$

Substituting Eq. (10) into Eq. (8), we derive a set of algebraic equations for $a_1, a_2, a_3, a_4, p_1, p_2, p_3, p_4, r_1, r_2, r_3, r_4$. Solving this system with the aid of Mathematica, we obtain eleven sets of solutions as follows:

Case 1

$$a_1 = a_2 = a_3 = 0, \quad r_4 = -4p_4^3, \tag{12}$$

where a_4, p_4 are free parameters. Taking $a_4 = 1$ and substituting Eq. (12) into Eq. (11), we can obtain the following bright soliton solution of Eq. (4)

$$u(x,t) = p_4^2 sech^2(\xi_4), \tag{13}$$

where $\xi_4 = p_4(x - 4p_4^2 t)$.

Case 2

$$a_2 = a_3 = 0, \quad r_1 = 4ip_4^3, \quad r_4 = -4p_4^3, \quad p_1 = -ip_4, \tag{14}$$

where a_1, a_4 and p_4 are free parameters. Substituting Eq. (14) into Eq. (11) yields the solution

$$u(x,t) = p_4^2\left[1 - \left(\frac{-e^{-\xi_4} + a_4e^{\xi_4} + a_1\sinh(\xi_4)}{e^{-\xi_4} + a_4e^{\xi_4} + a_1\cosh(\xi_4)}\right)^2\right], \tag{15}$$

where $\xi_4 = p_4(x - 4p_4^2 t)$.

Case 3

$$a_1 = a_3 = 0, \quad r_2 = -4ip_4^3, \quad r_4 = -4p_4^3, \quad p_1 = -ip_4, \tag{16}$$

where a_2, a_4 and p_4 are free parameters. Substituting Eq. (16) into Eq. (11), we get

$$u(x,t) = \frac{p_4^2 e^{-\xi_4} + a_4 p_4^2 e^{\xi_4} - a_2 p_2^2 \sin(\xi_2)}{e^{-\xi_4} + a_2 \sin(\xi_2) + a_4 e^{\xi_4}} - \frac{(a_2 p_2 \cos(\xi_2) - p_4 e^{-\xi_4} + a_4 p_4 e^{\xi_4})^2}{(e^{-\xi_4} + a_2 \sin(\xi_2) + a_4 e^{\xi_4})^2} \tag{17}$$

where $\xi_2 = p_2(x - 4ip_4^2 t)$, $\xi_4 = p_4 x - 4p_4^3 t$. If we take $a_4 = 1$, then Eq. (17) is

$$u(x,t) = \frac{2p_4^2 \cosh(\xi_4) - a_2 p_2^2 \sin(\xi_2)}{2\cosh(\xi_4) + a_2 \sin(\xi_2)} - \left(\frac{2p_4 \sinh(\xi_4) + a_2 p_2 \cos(\xi_2)}{2\cosh(\xi_4) + a_2 \sin(\xi_2)}\right)^2. \tag{18}$$

If we take $a_4 = -1$, then Eq. (17) is

$$u(x,t) = \frac{-a_2 p_2^2 \sin(\xi_2) - 2p_4^2 \sinh(\xi_4)}{-2\sinh(\xi_4) + a_2 \sin(\xi_2)} - \left(\frac{a_2 p_2 \cos(\xi_2) - 2p_4 \cosh(\xi_4)}{-2\sinh(\xi_4) + a_2 \sin(\xi_2)}\right)^2. \tag{19}$$

Case 4

$$a_3 = 0, \quad r_1 = 4ip_4^3, \quad r_2 = -4ip_4^3, \quad r_4 = -4p_4^3, \quad p_1 = p_2 = -ip_4, \tag{20}$$

where a_1, a_2, a_4 and p_4 are free parameters. Substituting Eq. (20) into Eq. (11), we have

$$u(x,t) = p_4^2 \left[1 - \left(\frac{(a_1+2)\sinh(\xi_4) - ia_2 \cos(\xi_2)}{(a_1+2)\cosh(\xi_4) + a_2 \sin(\xi_2)}\right)^2\right], \tag{21}$$

where $a_4 = 1, \xi_2 = -ip_4(x + 4p_4^2 t)$ and $\xi_4 = p_4(x - 4p_4^2 t)$.

Case 5

$$a_2 = a_3 = a_4 = 0, \quad r_1 = 4p_1^3, \quad r_4 = -4ip_1^3, \quad p_4 = -ip_1, \tag{22}$$

where a_1, p_1 are free parameters. Substituting Eq. (22) into Eq. (11) yields

$$u(x,t) = -p_1^2 \left[1 + \left(\frac{ie^{i\xi_1} - a_1 \sin(\xi_1)}{e^{i\xi_1} + a_1 \cos(\xi_1)}\right)^2\right], \tag{23}$$

where $\xi_1 = p_1(x + 4p_1^2 t)$. If we take $a_1 = -1$, then Eq. (23) can be reduced to the following periodic wave solution:

$$u(x,t) = -p_1^2 cosec^2(\xi_1). \tag{24}$$

Case 6

$$a_1 = a_3 = a_4 = 0, \quad r_2 = 4p_2^3, \quad r_4 = 4ip_2^3, \quad p_4 = ip_2, \tag{25}$$

where a_2, p_2 are free parameters. Substituting Eq. (25) into Eq. (11), we can write the following solution:

$$u(x,t) = -p_2^2 \left[1 + \left(\frac{-ie^{-i\xi_2} + a_2\cos(\xi_2)}{e^{-i\xi_2} + a_2\sin(\xi_2)}\right)^2\right], \tag{26}$$

where $\xi_2 = p_2(x + 4p_2^2 t)$. If we take $a_2 = i$, then Eq. (26) can be reduced as follows:

$$u(x,t) = -p_2^2 \sec^2(\xi_2). \tag{27}$$

Case 7

$$a_3 = a_4 = 0, \quad r_1 = r_2 = 4p_2^3, \quad r_4 = -4ip_2^3, \quad p_1 = p_2, \quad p_4 = -ip_2, \tag{28}$$

where a_1, a_2, p_2 are free parameters. Using the results (28), we can write the exact solutions of Eq. (4) as follows:

$$u(x,t) = -p_2^2 \left[1 + \left(\frac{ie^{i\xi_2} - a_1\sin(\xi_2) + a_2\cos(\xi_2)}{e^{i\xi_2} + a_1\cos(\xi_2) + a_2\sin(\xi_2)}\right)^2\right], \tag{29}$$

where $\xi_2 = p_2(x + 4p_2^2 t)$. Also, Eq. (29) can be reduced to the following form:

$$u(x,t) = -p_2^2 \left[1 + \left(\frac{(i + a_2)\cos(\xi_2) - (1 + a_1)\sin(\xi_2)}{(i + a_2)\sin(\xi_2) + (1 + a_1)\cos(\xi_2)}\right)^2\right]. \tag{30}$$

Case 8

$$a_1 = a_2 = a_4 = 0, \quad r_3 = -4p_3^3, \quad r_4 = 4p_3^3, \quad p_4 = -p_3, \tag{31}$$

where a_3, p_3 are free parameters. Substituting Eq. (31) into Eq. (11), we obtain

$$u(x,t) = p_3^2 \left[1 - \left(\frac{e^{\xi_3} + a_3\sinh(\xi_3)}{e^{\xi_3} + a_3\cosh(\xi_3)}\right)^2\right], \tag{32}$$

where $\xi_3 = p_3(x - 4p_3^2 t)$. Rewriting Eq. (32), we have

$$u(x,t) = \frac{2a_3 p_3^2}{(e^{\xi_3} + a_3\cosh(\xi_3))^2}. \tag{33}$$

Case 9

$$a_1 = a_4 = 0, \quad r_2 = -4ip_3^3, \quad r_4 = -r_3 = 4p_3^3, \quad p_2 = ip_3, \quad p_4 = -p_3, \tag{34}$$

Figure 1: Bright soliton U_1, where $p_4 = 1$, $w(T) = T$; M-type soliton U_1, where $p_4 = 1$, $w(T) = T^2$

where a_2, a_3, p_3 are free parameters. Substituting (34) into Eq. (11) yields:

$$u(x,t) = p_3^2\left[1 - \left(\frac{e^{\xi_3} + ia_2\cosh(\xi_3) + a_3\sinh(\xi_3)}{e^{\xi_3} + ia_2\sinh(\xi_3) + a_3\cosh(\xi_3)}\right)^2\right], \tag{35}$$

where $\xi_3 = p_3(x - 4p_3^2 t)$.

Case 10

$$a_4 = 0,\ r_1 = r_2 = -4ip_3^3,\ r_3 = -4p_3^3,\ r_4 = 4p_3^3,\ p_1 = p_2 = ip_3,\ p_4 = -p_3, \tag{36}$$

where a_1, a_2, a_3 and p_4 are free parameters. From Eqs. (36) and (11), the exact solution of the KdV equation is obtained as:

$$u(x,t) = p_3^2\left[1 - \left(\frac{e^{\xi_3} + (a_1 + a_3)\sinh(\xi_3) + ia_2\cosh(\xi_3)}{e^{\xi_3} + (a_1 + a_3)\cosh(\xi_3) + ia_2\sinh(\xi_3)}\right)^2\right], \tag{37}$$

where $\xi_3 = p_3(x - 4p_3^2 t)$ and $\xi_4 = -\xi_3$.

Case 11

$$a_1 = a_2 = a_4 = 0, \quad p_3 = r_3 = 0, \quad r_4 = -p_4^3, \tag{38}$$

where a_3 and p_4 are free parameters. Substituting (38) into Eq. (11), we have

$$u(x,t) = -\frac{p_4^2 e^{-2\xi_4}}{(a_3 + e^{-\xi_4})^2} + \frac{p_4^2 e^{-\xi_4}}{a_3 + e^{-\xi_4}}, \tag{39}$$

where $\xi_4 = p_4(x - p_4^2 t)$. From Eq. (39)

$$u(x,t) = \frac{a_3 p_4^2 e^{-\xi_4}}{(a_3 + e^{-\xi_4})^2}. \tag{40}$$

From Cases 1-11 and Eqs. (2)-(3), we obtain a series of exact stochastic solutions of Eq. (1), which are simplified as follows:

$$U_1(X,T) = p_4^2 sech^2\varphi_1(X,T) + w(T),$$

Figure 2: Periodic soliton U_3, where $p_2 = p_4 = a_2 = 1$, $w(T) = T^2$; Periodic soliton U_4, where $p_2 = p_4 = a_2 = 1$, $w(T) = T^2$

Figure 3: Periodic soliton U_6, where $p_1 = 1$, $w(T) = T$

$$U_2(X,T) = p_4^2 \left[1 - \left(\frac{-e^{-\varphi_1(X,T)} + a_4 e^{\varphi_1(X,T)} + a_1 \sinh(\varphi_1(X,T))}{e^{-\varphi_1(X,T)} + a_4 e^{\varphi_1(X,T)} + a_1 \cosh(\varphi_1(X,T))} \right)^2 \right] + w(T),$$

$$U_3(X,T) = \frac{2p_4^2 \cosh(\varphi_1(X,T)) - a_2 p_2^2 \sin(\varphi_2(X,T))}{2\cosh(\varphi_1(X,T)) + a_2 \sin(\varphi_2(X,T))} - \left(\frac{2p_4 \sinh(\varphi_1(X,T)) + a_2 p_2 \cos(\varphi_2(X,T))}{2\cosh(\varphi_1(X,T)) + a_2 \sin(\varphi_2(X,T))} \right)^2 + w(T),$$

$$U_4(X,T) = \frac{-a_2 p_2^2 \sin(\varphi_2(X,T)) - 2p_4^2 \sinh(\varphi_1(X,T))}{-2\sinh(\varphi_1(X,T)) + a_2 \sin(\varphi_2(X,T))} - \left(\frac{a_2 p_2 \cos(\varphi_2(X,T)) - 2p_4 \cosh(\varphi_1(X,T))}{-2\sinh(\varphi_1(X,T)) + a_2 \sin(\varphi_2(X,T))} \right)^2 + w(T),$$

$$U_5(X,T) = p_4^2 \left[1 - \left(\frac{(a_1 + 2)\sinh(\varphi_1(X,T)) - i a_2 \cos(\varphi_3(X,T))}{(a_1 + 2\cosh(\varphi_1(X,T)) + a_2 \sin(\varphi_3(X,T)} \right)^2 \right] + w(T),$$

where

$$\varphi_1(X,T) = p_4 \Big(X - 6\int_0^T W(T')dT' - 4p_4^2 T \Big),$$

$$\varphi_2(X,T) = p_2 \Big(X - 6\int_0^T W(T')dT' - 4ip_4^3 T \Big),$$

and

$$\varphi_3(X,T) = -ip_4\Big(X - 6\int_0^T W(T')dT' + 4p_4^2T\Big).$$

Also, the solution $U_1(X,T)$ is the solution obtained by Wadati in [1].

$$U_6(X,T) = -p_1^2 cosec^2(\varphi_4(X,T)) + w(T),$$

where $\varphi_4(X,T) = p_1\Big(X - 6\int_0^T W(T')dT' + 4p_1^2T\Big)$.

$$U_7(X,T) = -p_2^2\sec^2(\varphi_5(X,T)) + w(T),$$

$$U_8(X,T) = w(T) - p_2^2\left[1 + \left(\frac{(i+a_2)\cos(\varphi_5(X,T) - (1+a_1)\sin(\varphi_5(X,T))}{(i+a_2)\sin(\varphi_5(X,T)) + (1+a_1)\cos(\varphi_5(X,T))}\right)^2\right],$$

where $\varphi_5(X,T) = p_2\Big(X - 6\int_0^T W(T')dT' + 4p_2^2T\Big)$.

$$U_9(x,t) = \frac{2a_3p_3^2}{\left(e^{\varphi_6(X,T)} + a_3\cosh(\varphi_6(X,T)\right)^2} + w(T),$$

$$U_{10}(X,T) = p_3^2\left[1 - \left(\frac{e^{\varphi_6(X,T)} + ia_2\cosh(\varphi_6(X,T)) + a_3\sinh(\varphi_6(X,T))}{e^{\varphi_6(X,T)} - ia_2\sinh(\varphi_6(X,T)) + a_3\cosh(\varphi_6(X,T))}\right)^2\right] + w(T),$$

$$U_{11}(X,T) = p_3^2\left[1 - \left(\frac{e^{\varphi_6(X,T)} + (a_1+a_3)\sinh(\varphi_6(X,T)) + ia_2\cosh(\varphi_6(X,T))}{e^{\varphi_6(X,T)} + (a_1+a_3)\cosh(\varphi_6(X,T)) + ia_2\sinh(\varphi_6(X,T))}\right)^2\right] + w(T),$$

where $\varphi_6(X,T) = p_3\Big(X - 6\int_0^T W(T')dT' - 4p_3^2T\Big)$.

$$U_{12}(X,T) = \frac{a_3p_4^2e^{\varphi_7(X,T)}}{(a_3 + e^{\varphi_7(X,T)})^2} + w(T),$$

where $\varphi_7(X,T) = p_4\Big(X - 6\int_0^T W(T')dT' - p_4^2T\Big)$.

4 Conclusion

In this paper, we used the multi-wave method for the stochastic KdV equation that different cases of exact solutions of four-wave type are obtained, such as M-type wave solution, periodic solitary wave solution, triangular periodic wave solution, etc. The majority of these results are very different to those in Ref. [1]. This shows that some exact solutions with different wave forms can be expressed by the same one solution of four-wave type. From these results, we can say that multi-wave method is useful to many stochastic nonlinear partial differential equations.

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