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The Modified New Iterative Method for Solving Linear and Nonlinear Klein-Gordon Equations

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Abstract

In [1] Varsha and Jaferi proposed an iterative method for solving nonlinear functional equations, viz. nonlinear Voltera integral equations, algebraic equations and system of ordinary differential equations. Then Sachin and Varsha [2] implemented this method to solve linear and nonlinear partial differential equations of integer and fractional order. The aim of this article is to propose an efficient modification of the iterative method given in [2] for finding the exact solutions of linear and nonlinear Klein Gordon equations. The proposed modification is easy to use and we obtained excellent performance in comparison with the classical methods of Adomian decompositon method [3–7], variational iteration method [7–11] and homotopy perturbation method [12] that have been traditionally used for finding the solution of Klein-Gordon equation.

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1 Introduction

Klein-Gordon equation is one of the most important mathematical models in quantum field theory, nonlinear optics and plasma physics. The Klein-Gordon equation appears in physics in linear and nonlinear forms. The Klein-Gordon equation has been extensively studied by using traditional methods, such as finite difference method, finite element method and collocation method. Backlund transformations and the inverse scattering method were also applied to handle the Klein-Gordon equation.The methods investigated the concepts of existence, uniqueness of the solution and the weak solution as well. The objectives of these studies were mostly focused on the determination of numerical solutions where a considerable volume of calculations is usually needed.

In [3–12], the Adominan decomposition method, variational iteration method and homotpoy perturbation method were applied to obtain the exact solutions of linear and nonlinear Klein-Gordon equation. In this article, we apply the new iterative method (NIM) Versha [2] to linear and nonlinear Klein-Gordon equations. The NIM does not converge for linear and nonlinear inhomogeneous Klein-Gordon equations. We propose an efficient modification of NIM to apply it to both linear and nonlinear inhomogeneous Klein-Gordon equation. The modification is slight but the obtained results show that the modified technique is practical and efficient. Moreover, the modified technique minimizes the amount of calculations introduced by Adomian decomposition method, variational iteration method and homotopy perturbation method.

2 New Iterative Method [2]

Consider the following general functional equation

$$u(\overline{x}) = f(\overline{x}) + N(u(\overline{x})) \tag{1}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function. $\overline{x} = (x_1, x_2, \cdots, x_n)$. We are looking for a solution u of (1)having the series form

$$u(\overline{x}) = \sum_{n=0}^{\infty} u_i(\overline{x}) \tag{2}$$

The nonlinear operator N can be decomposed as

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i})u_j - N(\sum_{j=0}^{i-1} u_j)\}. \tag{3}$$

From equations (2)and (3), equation (1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i})u_j - N(\sum_{j=0}^{i-1} u_j)\} \tag{4}$$

We define the recurrence relation

$$\begin{cases} u_0 = f \\ u_1 = N(u_0) \\ u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}) \end{cases} \tag{5}$$

Then

$$(u_1 + \cdots + u_m) = N(u_0 + \cdots + u_m) \tag{6}$$

and

$$\sum_{i=0}^{\infty} u_i = f + N(\sum_{i=0}^{\infty} u_i) \tag{7}$$

The $k-$term approximate solution of (1) is given by $u = u_0 + u_1 + \cdots + u_{k-1}$

3 Modified New Iterative Method *(MNIM)*

The MNIM is based on including particular terms of the source term of inhomogeneous klein-Gordon equation into the integral representing $N(u)$ in NIM [2]. This selection is based on the following rules:

(i) If the source term is function of the independent variable, x only, we include it in $N(u)$.

(ii) If the source term is function of both independent variables x and t, we include it in $N(u)$.

(iii) If the source term contains the terms which are functions of x, t and both x and t, then we include in $N(u)$ the terms involving t and both x and t.

(iv) If the source term is $sin(x)sin(t)$, then NIM [2] can be applied to obtain the exact solution.

The MNIM will be demonstrated by applying it to various models of Klein-Gordon equation.

3.1 The Homogeneous and Inhomogeneous Linear Klein-Gordon Equations

Here the NIM [2] is applied to homogeneous and inhomogeneous Klein-Gordon equations. The obtained results show the excellent performance of the method.

Example 3.1 *Consider the homogeneous linear Klein-Gordon equation,*

$$u_{tt} - u_{xx} + u = 0 \tag{8}$$

with initial conditions

$$u(x, 0) = 0, u_t(x, 0) = x \tag{9}$$

The equation (8) is equivalent to the following integral equation

$$u = xt + \int_0^t \int_0^t (u_{xx} - u)dtdt$$

Set $u_0 = xt$ and $N(u) = \int_0^t \int_0^t (u_{xx} - u)dtdt$. Following the algorithm (5), the successive approximations are
$u_1 = N(u_0) = -\frac{1}{6}t^3x$
$u_2 = N(u_0 + u_1) - N(u_0) = \frac{1}{120}t^5x$
$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = -\frac{1}{5040}t^7x$
$\vdots$
Hence the series solution of (8) is given by

$$u(x,t) = \sum_0^\infty u_i = x(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\cdots) = x\sin(t)$$

which is exact solution.

Example 3.2 *Consider the homogeneous linear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u = 0 \tag{10}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = \cosh(x) \tag{11}$$

Equation (10)is equivalent to the following integral equation,

$$u = u = t\cosh(x) + \int_0^t \int_0^t (u_{xx} - u)dtdt$$

Set $u_0 = t\cosh(x)$ and $N(u) = \int_0^t \int_0^t (u_{xx} - u)dtdt$. Following the algorithm (5), the successive approximations are
$u_0 = \cosh(x)t$
$u_1 = N(u_0) = 0$
$u_2 = N(u_0 + u_1) - N(u_0) = 0$
$u_3 = 0,$
$\vdots$
Thus, the series solution of (10) is given by

$$u(x,t) = \cosh(x)t$$

which is exact solution.

Example 3.3 *Consider the in homogeneous linear Klein-Gordon equation*

$$u_{tt} - u_{xx} - u = -\cos(x)\cos(t) \qquad (12)$$

with initial conditions

$$u(x,0) = \cos(x), u_t(x,0) = 0 \qquad (13)$$

Equation (12)is equivalent to the following integral equation,

$$u = \int_0^t \int_0^t (u_{xx} + u)dtdt + \cos(x)\cos(t)$$

Set $u_0 = \cos(x)\cos(t)$ and $N(u) = \int_0^t \int_0^t (u_{xx} + u)dtdt$. Following the algorithm (5), the successive approximations are
$u_0 = \cos(x)\cos(t)$
$u_1 = N(u_0) = \int_0^t \int_0^t (u_0)_{xx} + u_0) = 0$
$u_2 = N(u_0 + u_1) - N(u_0) = 0$
$\vdots$
Thus, the series solution of (12) is given by

$$u(x,t) = \sum_0^\infty u_i = \cos(x)\cos(t)$$

which is exact solution.

3.2 The Inhomogeneous Linear and Nonlinear Klein-Gordon Equations

Here, we apply the NIM [2] to inhomogeneous linear and nonlinear Klein-Gordon equations and demonstrate that it does not converge to exact solutions. The MNIM is then applied to find the exact solution of these equations.

Example 3.4 *Consider inhomogeneous linear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u = 2\sin(x) \qquad (14)$$

with initial conditions

$$u(x,0) = \sin(x), u_t(x,0) = 1 \qquad (15)$$

The exact solution of (14) is $u = \sin(x) + sin(t)$

(a) NIM

Equation (14) is equivalent to the following integral form

$$u = \sin(x) + t + t^2 \sin(x) + \int_0^t \int_0^t (u_{xx} - u)dtdt$$

Set $u_0 = \sin(x) + t + t^2 \sin(x)$ and $N(u) = \int_0^t \int_0^t (u_{xx} + u)dtdt$. Following the algorithm (5), the successive approximations are
$u_0 = \sin(x) + t + t^2 \sin(x)$,
$u_1 = N(u_0) = -\frac{1}{6}(\sin x)t^4 - \frac{1}{6}t^3 - (\sin x)t^2$,
$u_2 = N(u_0 + u_1) - N(u_0) = \frac{1}{90}(\sin x)t^6 + \frac{1}{120}t^5 + \frac{1}{6}(\sin x)t^4$
Now the 3-term approximate solution,

$$u_0 + u_1 + u_2 = \sin(x) + t + t^2 \sin(x) + \frac{1}{90}(\sin x)t^6 + \frac{1}{120}t^5 + \frac{1}{6}(\sin x)t^4$$

shows that the series $\sum_0^\infty u_i$ does not converge to the exact solution $u = \sin(x) + \sin(t)$

(b) MNIM

Integrating equation (14) from 0 to t twice we get

$$u = \sin(x) + t + \int_0^t \int_0^t (u_{xx} - u + 2\sin(x))dtdt$$

Note that we have included the source term, $2sinx$ in the integral that will represent $N(u)$. Set $u_0 = \sin(x) + t$ and $N(u) = \int_0^t \int_0^t (u_{xx} - u + 2\sin(x))dtdt$. Following the algorithm (5), the successive approximations are
$u_0 = \sin(x) + t$
$u_1 = N(u_0) = -\frac{1}{6}t^3$
$u_2 = N(u_0 + u_1) - N(u_0) = \frac{1}{120}t^5$
$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = -\frac{1}{5040}t^7$
$\vdots$
Hence, the series solution of (14) is

$$u(x,t) = \sum_0^\infty u_i = \sin(x) + (t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots) = \sin(x) + \sin(t)$$

which is exact solution.

Example 3.5 *Consider inhomogeneous nonlinear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u^2 = x^2t^2 \quad (16)$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = x \quad (17)$$

The exact solution of (16) is $u(x,t) = xt$

(a) NIM

Equation (16) is equivalent to the following integral form

$$u = xt + \frac{x^2t^4}{12} + \int_0^t \int_0^t (u_{xx} - u^2)dtdt$$

Set $u_0 = xt + \frac{x^2t^4}{12}$ and $N(u) = \int_0^t \int_0^t (u_{xx} - u^2)dtdt$. Following the algorithm (5), the successive approximations are

$u_0 = xt + \frac{x^2t^2}{12}$

$u_1 = N(u_0) = \int_0^t \int_0^t (u_0)_{xx} - u_0^2)dtdt = -\frac{1}{12\,960}t^{10}x^4 - \frac{1}{252}t^7x^3 + \frac{1}{180}t^6 - \frac{1}{12}t^4x^2$

Now the 2-term approximate solution,

$$u_0 + u_1 = xt - \frac{1}{12\,960}t^{10}x^4 - \frac{1}{252}t^7x^3 + \frac{1}{180}t^6$$

shows that the series $\sum_0^\infty u_i$ does not converge to the exact solution $u(x,t) = xt$

(b) MNIM

Integrating equation (16) from 0 to t twice we get

$$u = xt + \int_0^t \int_0^t (u_{xx} - u^2 + x^2t^2)dtdt$$

Note that we have included the source term, x^2t^2 in the integral that will represent $N(u)$. Set $u_0 = xt$ and $u_0 = xt$

$u_1 = N(u_0) = \int_0^t \int_0^t (u_0)_{xx} - u_0^2 + x^2t^2)dtdt = 0$

$u_0 + u_1 = xt$

$u_2 = N(u_0 + u_1) - N(u_0) = 0$

$u_3 = 0$

$\vdots$

Hence, the series solution of (16) is

$$u(x,t) = \sum_0^\infty u_i = xt$$

which is exact solution.

Example 3.6 *Consider inhomogeneous nonlinear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4 \tag{18}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = 0 \tag{19}$$

The exact solution of (18) is $u(x,t) = x^2t^2$

(a) NIM

Equation (18) is equivalent to the following integral form

$$u = \int_0^t \int_0^t (u_{xx} - u^2)dtdt + x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30}$$

Set $u_0 = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30}$ and $N(u) = \int_0^t \int_0^t (u_{xx} - u^2)dtdt$. Following the algorithm (5), the successive approximations are

$u_0 = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30}$

$u_1 = N(u_0) = -\frac{1}{163\,800}t^{14}x^8 + \frac{1}{11\,880}t^{12}x^4 - \frac{1}{1350}t^{10}x^6 - \frac{1}{3240}t^{10} + \frac{11}{840}t^8x^2 - \frac{1}{30}t^6x^4 + \frac{1}{6}t^4$

$\vdots$

Now the 2-term approximate solution,$u_0 + u_1 = -\frac{1}{163\,800}t^{14}x^8 + \frac{1}{11\,880}t^{12}x^4 - \frac{1}{1350}t^{10}x^6 - \frac{1}{3240}t^{10} + \frac{11}{840}t^8x^2 + t^2x^2$ shows that the series $\sum_0^\infty u_i$ does not converge to the exact solution $u(x,t) = x^2t^2$

(b) MNIM

Integrating equation (18) from 0 to t twice we get

$$u = \int_0^t \int_0^t (u_{xx} - u^2 - 2t^2 + x^4t^4)dtdt + x^2t^2$$

Note that we have included the terms $-2t^2$ and x^4t^4 of the source term in the integral that will represent $N(u)$. Set $u_0 = x^2t^2$ and $N(u) = \int_0^t \int_0^t (u_{xx} - u^2 - 2t^2 + x^4t^4)dtdt$

$u_1 = N(u_0) = \int_0^t \int_0^t (u_0)_{xx} - u_0^2 - 2t^2 + x^4t^4)dtdt = 0$

$u_2 = N(u_0 + u_1) - N(u_0) = 0$

$u_3 = 0$

$\vdots$

Hence, the series solution of (16) is

$$u(x,t) = \sum_0^\infty u_i = x^2t^2$$

which is exact solution.

4 Conclusion

The main aim of this article is to present an effective modification of NIM [2] as an alternative method for solving the linear and nonlinear Klein-Gordon equations. The method is validated by applying it to several physical models of Klein-Gordon equations. The modification is slight but the obtained results show that it has many advantages over the existing methods of Adomian decomposition method, variational iteration method and homotopy perturbation method that have been convensionally used for solving Klein-Gordon equations. It has also been witnessed that a few approximations can be used to achieve a high degree of accuracy.

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