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Toeplitz and Hankel Operators with Radial Symbol on the Bergman Space[1]

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Abstract

In this paper,we give some results concerning Hankel operators and characterize when the Hankel operator and Toeplitz operator commute on weighted Bergman space.

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1 Introduction and Preliminaries

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. For $\alpha > -1$, we denote by dA_α the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. For $1 \leq p < +\infty$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a Banach space. The weighted Bergman space A^2_α is the closed subspace of analytic functions in the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$. For each $z \in \mathbb{D}$, the application:$L_z : A^2_\alpha \longrightarrow \mathbb{C}$ is continuous and can be represented as $L_z(f) = \langle f, K^{(\alpha)}_z \rangle_\alpha$, where

$$K^{(\alpha)}_z(w) = \frac{1}{(1-w\bar{z})^{2+\alpha}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}(w\bar{z})^n, z, w \in \mathbb{D}.$$

This means that, if P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α, then P_α can be defined by

$$(P_\alpha f)(w) = \langle f, K^{(\alpha)}_w \rangle_\alpha = \int_{\mathbb{D}} f(z) \frac{1}{(1-\bar{w}z)^{2+\alpha}} dA_\alpha(z).$$

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Let $U : L^2(\mathbb{D}, dA_\alpha(z)) \rightarrow L^2(\mathbb{D}, dA_\alpha(z))$ be the unitary operator defined by $Uf(z) = \widetilde{f(z)} = f(\bar{z})$, where f belongs to $L^2(\mathbb{D}, dA_\alpha(z))$. Let g be in $L^2(\mathbb{D}, dA_\alpha(z))$, we define a bounded linear operator M_g on $L^2(\mathbb{D}, dA_\alpha(z))$ as: $M_g(f) = gf$.Then we can define the small Hankel operator $H_g : A^2_\alpha(\mathbb{D}) \rightarrow A^2_\alpha(\mathbb{D})$ as PUM_g.

The question that we are interested in is: when do Toeplitz and Hankel operators commute? In 1964, Brown and Halmos [1] gave the sufficient and necessary condition for two bounded Toeplitz operators T_φ and T_ψ commute on Hardy space. On Bergman space of the unit disk, the first result was obtained by Axler and Cuckovic [2] who characterized commuting Toeplitz operators with harmonic symbols. The situation with a general symbols is rather more complicated. With the help of Mellin transform, Cuckovic and Rao [3] studied Toeplitz operators with monomial symbols. Louhichi, Strouse and Zakariasy [4] gave necessary and sufficient conditions for the product of two Toeplitz operators to be a Toeplitz operator and some other results.

In this paper, we investigate the commutativity of Toeplitz operators and Hankel operators on the weighted Bergman space of the unit disk.

2 Some Basic Results

An operator that will arise in our study of Toeplitz operators is the Mellin transform, defined for any function $\varphi \in L^1([0,1], rdr)$, by the formula $\widehat{\varphi_\alpha}(z)$ is the Mellin transform :

$$\widehat{\varphi_\alpha}(z) = \int_0^1 \varphi(r)(1-r^2)^\alpha r^{z-1} dr.$$

which is a bounded holomorphic function in the half plane $\{z : Rez \geq 2\}$.

Let $\varphi \in L^1(\mathbb{D}, dA_\alpha)$ be a radial function, i.e. suppose that: $\varphi(z) = \varphi(|z|), z \in \mathbb{D}$. Then, if φ is a T-function, the Toeplitz operator with symbol φ acts in a very simple way. In fact, if we define the function φ_r on $[0,1]$ by $\varphi_r(s) = \varphi(s)$, then a direct calculation shows that:

$$\langle T_\varphi(z^k), z^l \rangle_\alpha = \begin{cases} 0 & for \quad k \neq l \\ 2(\alpha+1)\widehat{\varphi_\alpha}(2k+2) & for \quad k = l. \end{cases}$$

So that if $k \in \mathbb{N}$:

$$\begin{aligned} T_\varphi(z^k) &= 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} \int_0^1 \varphi r^{2k+1}(1-|r|^2)^\alpha dr z^k \\ &= 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} \widehat{\varphi_{\alpha,r}}(2k+2) z^k. \end{aligned}$$

Thus T_φ is a diagonal operator on A^2_α with coeffcient sequence

$$(2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}\widehat{\varphi_{\alpha,r}}(2k+2))_{k=0}^{\infty}.$$

This makes it relatively simple to work with the product of two operators with such radial symbols.

Now, we define the "radialization" of a function $f \in L^1(\mathbb{D}, dA_\alpha)$ by :

$$rad(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)dt.$$

It is clear that a function f is a radial if and only if $rad(f) = f$. This permits us to prove a very simple but essential theorem.

Let $\mathfrak{R}$ be the space of square integrable radial functions on $\mathbb{D}$. As before, we identify these functions with the associated functions on $[0,1]$ that are square integrable with respect to rdr measure. By using that trigonometric polynomials are dense in $L^2(\mathbb{D}, dA_\alpha)$ and that, for $k_1 \neq k_2$, $e^{ik_1\theta}R$ is orthogonal to $e^{ik_2\theta}R$, we see that:

$$L^2(\mathbb{D}, dA_\alpha) = \oplus_{k\in\mathbb{Z}} e^{ik\theta} R^\alpha.$$

Definition 2.1 *Let φ be a function in $L^1(\mathbb{D}, dA_\alpha)$ which is of the form $e^{ik\theta}f$ where f is a radial function. Then we say that φ is a quasihomogeneous function of quasihomogeneous degree k.*

A direct calculation gives the following lemma which we shall use often.

Lemma 2.2 *Let $k, p \in \mathbb{Z}_+$ and let φ be an integrable radial function, then,*

$$T_{e^{ip\theta}\varphi}(z^k) = 2(\alpha+1)\frac{\Gamma(k+p+2+\alpha)}{(k+p)!\Gamma(2+\alpha)}\widehat{\varphi_\alpha}(2k+p+2)z^{k+p},$$

and

$$T_{e^{-ip\theta}\varphi}(z^k) = \begin{cases} 0 & if \quad k < p \\ 2(\alpha+1)\frac{\Gamma(k-p+2+\alpha)}{(k-p)!\Gamma(2+\alpha)}\widehat{\varphi_\alpha}(2k-p+2)z^{k-p} & if \quad k \geq p. \end{cases}$$

Lemma 2.3 *Let $k, p \in \mathbb{Z}_+$ and let φ be an integrable radial function, then,*

$$H_{e^{ip\theta}\varphi}(z^k) = 0$$

and

$$H_{e^{-ip\theta}\varphi}(z^k) = \begin{cases} 0 & if \quad k > p \\ 2(\alpha+1)\frac{\Gamma(-k+p+\alpha+2)}{(-k+p)!\Gamma(2+\alpha)}\widehat{\varphi_\alpha}(p+2)z^{-k+p} & if \quad k \leq p. \end{cases}$$

3 Commutativity of Toeplitz and Hankel operators

In this section we study the commutativity of Toeplitz operators and Hankel operators on the Bergman space with quasihomogeneous symbols or radial function.

Theorem 3.1 *Let φ be a bounded radial function and $e^{-ip\theta}\psi$ a quasihomogeneous bounded function of degree $-p<0$, then $T_\varphi H_{e^{-ip\theta}\psi} = H_{e^{-ip\theta}\psi}T_\varphi$ if and only if one of the following conditions is satisfied*

(1) $\widehat{\psi}(p+2)=0$;

(2) For $0\le k\le p$, $\frac{\Gamma(-k+p+\alpha+2)}{(-k+p_2)!\Gamma(\alpha+2)}\widehat{\varphi}(-2k+2p+2)=\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\widehat{\varphi}(2k+2)$.

Proof. For $k\ge 0$, we have

$$T_\varphi H_{e^{-ip\theta\psi}}(z^k)=\begin{cases}4(\alpha+1)^2[\frac{\Gamma(-k+p+\alpha+2)}{(-k+p)!\Gamma(\alpha+2)}]^2\widehat{\psi}(p+2) & \\ \qquad\cdot\widehat{\varphi}(-2k+2p+2)z^{-k+p} & if \quad p\ge k\ge 0\\ 0 & if \quad k>p.\end{cases}$$

and

$$H_{e^{-ip\theta\psi}}T_\varphi(z^k)=\begin{cases}4(\alpha+1)^2\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\frac{\Gamma(-k+p+\alpha+2)}{(-k+p)!\Gamma(\alpha+2)}\widehat{\psi}(p+2) & \\ \qquad\cdot\widehat{\varphi}(2k+2)z^{-k+p} & if \quad p\ge k\ge 0\\ 0 & if \quad k>p.\end{cases}$$

If $T_\varphi H_{e^{-ip\theta}\psi} = H_{e^{-ip\theta}\psi}T_\varphi$, we have, for $0\le k\le p$.

$$\frac{\Gamma(-k+p+\alpha+2)}{(-k+p)!\Gamma(\alpha+2)}\widehat{\varphi}(-2k+2p+2)=\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\widehat{\varphi}(2k+2).$$

Theorem 3.2 *Let ψ be a bounded radial function and $e^{ip\theta}\varphi$ a quasihomogeneous bounded function of degree $p>0$, then $T_{e^{ip\theta}\varphi}H_\psi = H_\psi T_{e^{ip\theta}\varphi}$ if and only if $\widehat{\psi}(2)=0$ or $\widehat{\varphi}(p+2)=0$.*

Proof. For $k\ge 0$, we have

$$T_{ip\theta\varphi}H_\psi(z^k)=\begin{cases}4(\alpha+1)^2\frac{\Gamma(p+\alpha+2)}{p!\Gamma(\alpha+2)}\widehat{\psi}(2)\widehat{\varphi}(p+2)z^p & if \quad k=0\\ 0 & if \quad k>0.\end{cases}$$

and

$$H_\psi T_{e^{ip\theta}\varphi}(z^k) = 2(\alpha+1)\frac{\Gamma(k+p+\alpha+2)}{(k+p)!\Gamma(\alpha+2)}\widehat{\varphi}(2k+2p+2)H_\psi(z^{k+p}) = 0.$$

Then we have, $T_{e^{ip\theta}\varphi}H_\psi = H_\psi T_{e^{ip\theta}\varphi}$ if and only if $(\alpha+1)^2\frac{\Gamma(p+\alpha+2)}{p!\Gamma(\alpha+2)}\widehat{\psi}(2)\widehat{\varphi}(p+2)z^p = 0$, that is $\widehat{\psi}(2) = 0$ or $\widehat{\varphi}(p+2) = 0$. The converse is easy to get.

Theorem 3.3 *Let ψ be a bounded radial function and $e^{-ip\theta}\varphi$ a quasihomogeneous bounded function of degree $-p < 0$, then $T_{e^{-ip\theta}\varphi}H_\psi = H_\psi T_{e^{-ip\theta}\varphi}$ if and only if $\widehat{\psi}(2) = 0$ or $\widehat{\varphi}(p+2) = 0$.*

Proof. For $k \geq 0$, we have $T_{-ip\theta\varphi}H_\psi(z^k) = 0$, since

$$T_{-ip\theta\varphi}H_\psi(z^k) = \begin{cases} 2(\alpha+1)\widehat{\psi}(2)T_{-ip\theta\varphi}z^0 & if \quad k = 0 \\ 0 & if \quad k > 0. \end{cases}$$

and

$$H_\psi T_{-ip\theta\varphi}(z^k) = \begin{cases} \frac{2(\alpha+1)\Gamma(k-p+\alpha+2)}{(k-p)!\Gamma(\alpha+2)}\widehat{\varphi}(2k-p+2)H_\psi(z^{k-p}) & if k \geq p \\ 0 & if p-1 \geq k \geq 0. \end{cases}$$

$$= \begin{cases} 4(\alpha+1)^2\widehat{\varphi}(p+2)\widehat{\psi}(2) & if \quad k = p \\ 0 & if \quad k \neq p. \end{cases}$$

Then we have $T_{e^{ip\theta}\varphi}H_\psi = H_\psi T_{e^{ip\theta}\varphi}$ if and only if $\widehat{\psi}(2) = 0$ or $\widehat{\varphi}(p+2) = 0$.

It is easy to get the converse is true.

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