Applied Mathematical Sciences, Vol. 6, 2012, no. 59, 2945 - 2955

# Ruled Surfaces with Different Blaschke Approach

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**Abstract**

In this paper, using dual elements, Frenet and Blaschke frames, unit dual spherical curves are studied with ruled surfaces. Then, from some well-known approaches, new Blaschke approach of ruled surfaces is given. Moreover, kinematic interpretation of the moving Blaschke frame is presented with some theorems and their proofs. Finally, when we take the derivatives of the indicatrices by changing the parameter of the curve, we have new results.

**Mathematics Subject Classification:** 53A04, 53A05, 53A17

**Keywords:** Ruled surface, dual vectors, Frenet frame

# 1 Introduction

Dual spherical geometry, expressed with the help of dual unit vectors, is closely analogous to real spherical geometry, expressed with the help of real unit vectors. Therefore the properties of elementary real spherical geometry can also be carried over by analogy into the geometry of lines in $R^3$. A differentiable curve on the dual unit sphere, depending on a real parameter $t$, represents a differentiable family of straight lines in $R^3$ which we call a "ruled surface" [3,4]

As it is known the oriented lines in $R^3$ are in one-to-one correspondance with the points of the dual unit sphere $ID^3$ (E. Study). Using this correspondance, we extend the ruled surfaces with different Blaschke approach. At this time, [8] shows us that we can study a ruled surface as a curve on the dual unit sphere by using Blachke approach. From [6,7], we can see that a dual curve can be defined as the set of dual points.

In addition, we present the kinematic interpretation of dual representations with Blaschke approach. At this time, from the relations between Blaschke, Frenet frames and Darboux vector in [12,13], we find and show different Blaschke approach.

# 2 BASIC CONCEPTS

Now we give basic concepts on classical differential geometry of space curves. References [1, 2, 3, 4, 5, 6, 7, 9, 10] contain basic concepts about the dual elements and one to one correspondance between ruled surface and dual spherical curves.

## 2.1 Dual Elements

If $a$ and $a^*$ are real numbers and $\varepsilon^2 = 0$, the combination

$$\widehat{A} = a + \varepsilon a^* \tag{1}$$

is called a dual number. The dual number system is a "complex" system with 2 units just as in ordinary complex numbers $\varepsilon$ is the dual unit and has the properties $\varepsilon \neq 0, 0\varepsilon = 0, 1\varepsilon = \varepsilon 1 = \varepsilon, \varepsilon^2 = 0$.

Given dual numbers $\widehat{A} = a + a^*$ and $\widehat{B} = b + \varepsilon b^*$ the rules for combination can be defined as:

$$\begin{aligned} Equality &: A = B \Longleftrightarrow a = b, a^* = b^* \\ Addition &: A + B \Leftrightarrow (a+b) + \varepsilon(a^* + b^*) \\ Multiplication &: AB \Longleftrightarrow ab + \varepsilon(a^* b + ab^*) \end{aligned} \tag{2}$$

Division isn't always possible. Only division by $b + \varepsilon b^*, b \neq 0$,is possible and unambigious. The division of dual number is defined as:

$$Division : \frac{A}{B} = \frac{a}{b} + \varepsilon(\frac{a^* b - ab^*}{b^2}), b \neq 0. \tag{3}$$

## 2.2 Frenet Frame

We assume that the curve $\alpha$ is parametrized by arclength. Then, $\alpha'(s)$ is the unit tangent vector to the curve, which we denote by $T(s)$ Since t has constant length, $T'(s)$ will be orthogonal to $T(s)$. If $T'(s) \neq 0$ then we define principal normal

$$N(s) = \frac{T'(s)}{\|T'(s)\|} \tag{4}$$

vector and the curvature

$$\kappa(s) = \|T'(s)\|$$

So far, we have

$$T'(s) = \tau(s)N(s) \tag{5}$$

If $\kappa(s) = 0$, the principal normal vector is not defined. If $\kappa(s) \neq 0$ then the binormal vector $B(s)$ is given by

$$B(s) = T(s) \times N(s)$$

Then $\{T(s), N(s), B(s)\}$ form a right-handed orthonormal basis for $R^3$. In summary Frenet formulas can be given as

$$\begin{aligned} T'(s) &= \kappa(s)N(s) && (6)\\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) && (7)\\ B'(s) &= -\tau(s)N(s). && (8) \end{aligned}$$

On the other hand, by assuming that the unit tangent indicatrices of the curve $\alpha$ is parametrized by $s_T$, we can give

$$\begin{aligned} \frac{dT}{ds_T} &= \frac{dT}{ds}\frac{ds}{ds_T} && (9)\\ &= (\kappa N)\frac{ds}{ds_T} \\ &= N. \end{aligned}$$

Because,

$$\begin{aligned} \frac{ds}{ds_T} &= \frac{1}{\kappa} && (10)\\ ds_T &= \kappa ds \\ s_T &= \int \kappa ds \end{aligned}$$

Now, we can investigate the variaton of Frenet Frame for $s_T$. If we write $\dot{T}$ , $\dot{N}$ and $\dot{B}$ as:

$$\dot{T} = \frac{dT}{ds}\frac{ds}{ds_T} \tag{11}$$

$$\begin{aligned}
&= (\kappa N)\frac{1}{\kappa} \\
&= N, \\
\dot{N} &= \frac{dN}{ds_T} \\
&= \frac{dN}{ds}\frac{ds}{ds_T} \\
&= \frac{1}{\kappa}(-\kappa T + \tau B) \\
&= -T + \frac{\tau}{\kappa}B, \\
\dot{B} &= \frac{dB}{ds_T} \\
&= \frac{dB}{ds}\frac{ds}{ds_T} \\
&= \frac{1}{\kappa}(-\tau N) = -\frac{\tau}{\kappa}N
\end{aligned}$$

then we are able to write the frame

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \frac{\tau}{\kappa} \\ 0 & -\frac{\tau}{\kappa} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{12}$$

and also we can give Darboux vector,

$$W = \frac{\tau}{\kappa}T + B \tag{13}$$

then

$$\dot{T} = W \wedge T, \tag{14}$$
$$\dot{N} = W \wedge N \tag{15}$$
$$\dot{B} = W \wedge B \tag{16}$$

## 2.3 Blaschke Frame

Let $M(s,u)$ be the ruled surface and $A(s)$ be the dual spherical curve in $ID^3$;

$$A(s) = a(s) + \varepsilon a^\star(s) \tag{17}$$

We now define an orthonormal moving frame along this dual curve as follows:

$$A_1 = A(s), \tag{18}$$
$$A_2 = \frac{A_1'}{\|A_1'\|}, \tag{19}$$
$$A_3 = A_1 \wedge A_2. \tag{20}$$

# 3 DUAL CURVES AND RULED SURFACES

## 3.1 UNIT DUAL SPHERICAL CURVES AND RULED SURFACES [13]

In this section we can investigate unit dual spherical curves with ruled surfaces in these kinds:

At first, $A(s) = a(s) + \varepsilon a^*(s)$ unit dual curve that corresponds to ruled surface

$$\begin{aligned} \Phi(s,u) &= a(s) \wedge \alpha(s) + ua(s) \\ &= \alpha(s) + ua(s) \end{aligned} \tag{21}$$

where $\alpha(s)$ is a base curve and $a(s)$ is a rulling of ruled surface.

Secondly, $A(s) = a(s) + \varepsilon a^*(s)$ unit dual curve that corresponds to ruled surface

$$\Phi(s,u) = \alpha(s) + ua(s) \tag{22}$$

where

$$a^*(s) = \alpha(s) \wedge a(s)$$

**Example 1.** $A(s) = a(s)+\varepsilon a^*(s)$ is a unit dual curve where $a(s) = (0, 0, 1)$ and $a^*(s) = (\sin s, -\cos s, 0)$. $A(s)$ is corresponding to ruled surface $\Phi(s,u)$ :

$$\begin{aligned} \Phi(s,u) &= a(s) \wedge a^*(s) + ua(s) \\ &= (\cos s, \sin s, 0) + u(0, 0, 1) \end{aligned} \tag{23}$$

This shows that we can get a cylinder. On the contrary, if we take $\Phi(s,u)$ ruled surface as follows

$$\Phi(s,u) = (\cos s, \sin s, 0) + u(0, 0, 1) \tag{24}$$

then $A(s) = a(s) + \varepsilon a^*(s)$ is a unit dual curve where

$$\begin{aligned} a(s) &= (0, 0, 1) \\ a^*(s) &= \alpha(s) \wedge a(s) \\ &= (\sin s, -\cos s, 0). \end{aligned} \tag{25}$$

## 3.2 DUAL CURVES AND RULED SURFACES WITH BLASCHKE FRAME

Let

$$\begin{aligned} \alpha \ &: \ I \to E^3 \\ &\quad\ s \to \alpha(s) \end{aligned} \tag{26}$$

be unit speed curve and $\{T, N, B\}$ Frenet frame of $\alpha$. $T, N, B$ are the unit tangent, principal, normal and binormal vectors respectively. With the assistance of $\alpha$, we define a dual curve in $ID^3$. So, let us have a closed spherical dual curve $\widehat{\alpha}$ of class $C^1$ on a unit dual sphere $S^2_{ID}$in $ID^3$. The curve $\widehat{\alpha}$ describes a closed dual spherical motion.

Here we can easily say that curve $\widehat{\alpha}$ on unit dual sphere is corresponding to a ruled surface in $E^3$.

On the other hand, we can give the ruled surfaces of $\alpha$ that produced by $T, N, B$ as follows in respectively,

$$\Phi_T(s, v) = \alpha(s) + \nu T(s) \tag{27}$$

with dual curve representation

$$\widehat{\alpha}(s) = \widehat{T}(s) = T(s) + \varepsilon\alpha \wedge T(s). \tag{28}$$

And for $N$ with dual curve representation

$$\begin{aligned} \Phi_N(s, v) &= \alpha(s) + \nu N(s), \\ \widehat{N}(s) &= N(s) + \varepsilon\alpha \wedge N(s), \end{aligned} \tag{29}$$

and then for $B$ with dual curve representation

$$\begin{aligned} \Phi_B(s, v) &= \alpha(s) + \nu B(s), \\ \widehat{B}(s) &= B(s) + \varepsilon\alpha \wedge B(s). \end{aligned} \tag{30}$$

In [7,8], it can be seen that a dual curve can be defined as the set of dual points. So they choose $\{T, N\}$ as points. According to all of these, now we define an orthonormal moving frame along dual curve as follows in $ID^3$;The tangent indicatrice of $\widehat{\alpha}$ is

$$A_1(s) = \widehat{T}. \tag{31}$$

The principal normal indicatrice of $\widehat{\alpha}$ is

$$A_2(s) = \widehat{N} = \frac{\frac{d\widehat{T}}{ds}}{\left\| \frac{d\widehat{T}}{ds} \right\|}, \tag{32}$$

The binormal indicatrice of $\widehat{\alpha}$ is

$$A_3(s) = \widehat{B} = \widehat{T} \times \widehat{N} \tag{33}$$

Thus, we get a frame on dual sphere and the following result can be given as:

**RESULT:** $\left\{\widehat{T}, \widehat{N}, \widehat{B}\right\}$ is a Blaschke frame. Indeed,

$$\begin{aligned} A_1 &= \widehat{T}, \\ A_2 &= \widehat{N} = \frac{\frac{d\widehat{T}}{ds}}{\left\|\frac{d\widehat{T}}{ds}\right\|}, \\ A_3 &= \widehat{B} = A_1 \times A_2 \end{aligned} \tag{34}$$

Now, we investigate the motion of this Blaschke frame:

$$\begin{bmatrix} \widehat{T}' \\ \widehat{N}' \\ \widehat{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau+\varepsilon \\ 0 & -\tau-\varepsilon & 0 \end{bmatrix} \begin{bmatrix} \widehat{T} \\ \widehat{N} \\ \widehat{B} \end{bmatrix} \tag{35}$$

At this time, it can be given with its dual indicatrice form as follows, too:

$$\begin{aligned} \frac{d\widehat{T}}{ds_T} &= \dot{\widehat{T}} \\ &= \frac{d\widehat{T}}{ds}\frac{ds}{ds_T} \\ &= \frac{1}{\kappa}(\kappa\widehat{N}) = \widehat{N} \end{aligned} \tag{36}$$

and also

$$\begin{aligned} \frac{d\widehat{N}}{ds_T} &= \dot{\widehat{N}} \\ &= \frac{1}{\kappa}(-\kappa\widehat{T} + (\tau+\varepsilon)\widehat{B}), \\ \frac{d\widehat{B}}{ds_T} &= \dot{\widehat{B}} \\ &= \frac{-\tau-\varepsilon}{\kappa}\widehat{N} \end{aligned} \tag{37}$$

In this way, the motion of this system is

$$\begin{bmatrix} \dot{\widehat{T}} \\ \dot{\widehat{N}} \\ \dot{\widehat{B}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \frac{\tau+\varepsilon}{\kappa} \\ 0 & -\frac{\tau+\varepsilon}{\kappa} & 0 \end{bmatrix} \begin{bmatrix} \widehat{T} \\ \widehat{N} \\ \widehat{B} \end{bmatrix} \tag{38}$$

Subsequently, as [8,12-13], we can define Darboux screw as follows:

$$\begin{aligned} \dot{\widehat{T}} &= \widehat{W} \wedge \widehat{T} \\ \dot{\widehat{N}} &= \widehat{W} \wedge \widehat{N} \\ \dot{\widehat{B}} &= \widehat{W} \wedge \widehat{B} \end{aligned} \tag{39}$$

Then, we get

$$d_F X = \widehat{W} \wedge X \tag{40}$$

from [6]. And with all of these, we give the Darboux screw as:.

$$\widehat{W} = (\frac{\tau+\varepsilon}{\kappa})\widehat{T}(t) + \widehat{B}(t) \tag{41}$$

in [8,12-13].

On the other hand, if we take $\dot{T} = \widehat{T}$, $\dot{N} = \widehat{N}$, $\dot{B} = \widehat{B}$, then the Blaschke's inviriants of the dual curve $\widehat{T}(s)$ is given by

$$\begin{aligned} P &= p + \varepsilon p^\star = \|T'\| = \kappa \\ Q &= q + \varepsilon q^\star = \frac{\det\left\|\widehat{T}, \widehat{T}', \widehat{T}''\right\|}{P^2} \end{aligned} \tag{42}$$

where

$$\kappa = P \tag{43}$$

$$\tau + \varepsilon = Q \tag{44}$$

Thus, the distribution parameters of the ruled surfaces $A_1, A_2, A_3$ respectively can be given as:

$$\begin{aligned} \lambda_1 &= \frac{p^*}{p} \\ \lambda_2 &= \frac{pp^* + qq^*}{p^2 + q^2} \\ \lambda_3 &= \frac{q^*}{q} \end{aligned} \tag{45}$$

where

$$\begin{aligned} p &= \kappa,\ p^* = 0; \\ q &= \tau, q^* = 1. \end{aligned} \tag{46}$$

In this situation, we can give $\lambda_1, \lambda_2, \lambda_3$ as follows:

$$\begin{aligned} P_{\dot{T}} &= \lambda_1 = 0 \\ P_{\dot{N}} &= \lambda_2 = \frac{0+\tau}{\kappa^2+\tau^2} = \frac{\tau}{\kappa^2+\tau^2} \\ P_{\dot{B}} &= \lambda_3 = \frac{1}{\tau} \end{aligned} \tag{47}$$

and the distribution parametrs of the ruled surfaces are:

$$\begin{aligned} P_T &= \frac{\det(T,T,T')}{\|A_1'\|^2} = 0 \\ P_N &= \frac{\det(T,N,N')}{\|N'\|^2} = \frac{\det(T,N,-\kappa T+\tau B)}{\kappa^2+\tau^2} \\ &= \frac{\tau}{\kappa^2+\tau^2} \\ P_B &= \frac{\det(T,B,B')}{\|B'\|^2} = \frac{1}{\tau} \end{aligned} \tag{48}$$

### 3.2.1 Kinematic Interpretation

In this section, from [8], we can give the kinematic interpretation of the moving Blaschke frame which is provided by the Blaschke invariants $\kappa$ and $\tau+\varepsilon$. Smiliarly, here we show the Frenet curvatures of the curve $\alpha$ with $\kappa(s)$ and $\tau(s)$; Bishop curvatures of the curve $\alpha$ with $\kappa(s)$ and $\tau(s)$.

$$\widetilde{W} = a + \varepsilon a^* = (\frac{\tau+\varepsilon}{\kappa})\widehat{T} + \widehat{B}$$

known as dual Darboux vector, the dual angular velocity vector of the Blaschke frame with respect to itself has a component $(\frac{\tau+\varepsilon}{\kappa})$ about $\widehat{T}$

$$\begin{aligned} \left\|\widetilde{W}\right\| &= \left\langle \widetilde{W}, \widetilde{W} \right\rangle = \sqrt{1+(\frac{\tau+\varepsilon}{\kappa})^2} = \sqrt{\frac{\kappa^2+\tau^2}{\kappa^2} + 2\varepsilon(\frac{\tau}{\kappa})} \\ &= \sqrt{\frac{\kappa^2+\tau^2}{\kappa^2}} + \varepsilon\frac{\frac{\tau}{\kappa}}{\sqrt{\frac{\kappa^2+\tau^2}{\kappa^2}}} \\ &= w + \varepsilon w^* \end{aligned}$$

is angular speed of $\widehat{T}$ about $\widetilde{W}$;

In this way, we can give these theorems:

**Theorem 3.1** *The pitch of instantaneous screw motion is*

$$\begin{aligned} \frac{w^*}{w} &= \frac{\frac{\tau}{\kappa}}{1+(\frac{\tau}{\kappa})^2} \\ &= \frac{H}{1+H^2} \end{aligned} \tag{49}$$

*as [8]. Here,* $H = \frac{\tau}{\kappa}$ *is a harmonic curvature.*

**Proof.**

$$\begin{aligned}\left\|\widetilde{W}\right\| &= \sqrt{1+(\frac{\tau+\varepsilon}{\kappa})^2} = \sqrt{\frac{\kappa^2+\tau^2}{\kappa^2}+2\varepsilon(\frac{\tau}{\kappa})} && (50)\\ &= \sqrt{\frac{\kappa^2+\tau^2}{\kappa^2}}+\varepsilon\frac{\frac{\tau}{\kappa}}{\sqrt{\frac{\kappa^2+\tau^2}{\kappa^2}}} \\ &= w+\varepsilon w^* && (51)\end{aligned}$$

from the pitch of $\widehat{T}$ along $\widetilde{W}$ ,

$$\frac{w^*}{w} = \frac{\frac{\tau}{\kappa}}{\frac{\kappa^2+\tau^2}{\kappa^2}} = \frac{\frac{\tau}{\kappa}}{1+(\frac{\tau}{\kappa})^2} \quad (52)$$

**Theorem 3.2** *The pitch of instantaneous screw motion is constant if and only if the curve $\alpha$ is helix.*

**Proof.** If $\frac{w^*}{w}$ is constant

$$\frac{w^*}{w} = \text{constant}$$

then $H$ is constant, then $\frac{\tau}{\kappa}$ is constant, then the curve $\alpha$ is helix.

On the contrary, if the curve $\alpha$ is helix, then $\frac{\tau}{\kappa}$ is constant and then $\frac{H}{1+H^2}$ is constant, so

$$\frac{w^*}{w} = \text{constant}.$$

# 4 CONCLUSIONS

In this study, we have seen that the curve on unit dual sphere is corresponding to a ruled surface in $E^3$. After giving an orthonormal moving frame along dual curve with its tangent, principal normal and binormal indicatrices, we have calculated the distribution parameters of the ruled surfaces. Finally, it is shown that the pitch of instantaneous screw motion is constant if and only if the curve is helix.

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**Received: January, 2012**