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Partition of a Nonempty Fuzzy Set in Nonempty Convex Fuzzy Subsets

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Abstract

This paper presents the concept of partition of a nonempty fuzzy set in nonempty convex fuzzy subsets. We restrict our discussion when the universal set, where are defined the fuzzy sets, is the real n-dimensional Euclidean space. We also show that a nonempty fuzzy set has a least one partition in nonempty convex fuzzy subsets.

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1 Introduction

Standard books of mathematics [1], [2], [4] define a partition of a classical set A as a family of pairwise disjoint nonempty subsets such that their union yields the original set. When the universal set X, where the sets are formulated, is the real n-dimensional Euclidean space $\mathbb{R}^n$, is possible that its subsets have properties such as *convexity*, which establishes that given two elements in a subset, all the points on the straight-line connecting the two points are also in the subset.

Zadeh [6] defined a *fuzzy set* A over a universal set X by using its *membership function* $f_A(x)$, which assigns to every element $x \in X$ a real number in $[0, 1]$, i.e., $f_A : X \mapsto [0, 1]$. He also extended the definition of convexity to fuzzy sets when the universal set X is $\mathbb{R}^n$.

The aim of this paper is to show that under the definition of fuzzy set, which was given by Zadeh, and the extension of convexity to fuzzy sets, is also possible to have at least one classical partition of a fuzzy set in convex fuzzy subsets, i.e., a family of pairwise disjoint nonempty convex fuzzy subsets such that their union yields the original fuzzy set.

This paper is organized as follows: Section 2 gives some preliminaries in order to establish the mathematical background, section 3 gives the definition of partition of a nonempty fuzzy set in nonempty convex fuzzy subsets and a simple result, finally, section 4 gives the conclusion.

2 Preliminaries

Definition 2.1 (Fuzzy set) [2], [4] *Given a universal set X, a fuzzy set A is defined by a function of the form $f_A : X \mapsto [0,1]$*

For a fuzzy set A, the function $f_A : X \mapsto [0,1]$ is called *membership function*, and the value $f_A(x)$ is called the *degree of membership* of x in the fuzzy set A. A fuzzy set A is a subset of a fuzzy set B, written as $A \subseteq B$, if and only if $f_A(x) \leq f_B(x)$ for all $x \in X$. Two fuzzy sets A and B are *equal*, written as $A = B$, if and only if $f_A(x) = f_B(x)$ for all $x \in X$. A fuzzy set is *empty* if and only if its membership function is identically zero on X, i.e., $f_\emptyset(x) = 0$ for all $x \in X$. The *union* of two fuzzy sets A and B is a fuzzy set C, written as $C = A \cup B$, whose membership function is $f_C(x) = \max\{f_A(x), f_B(x)\}$ for all $x \in X$, or in abbreviated form $f_C(x) = f_A(x) \vee f_B(x)$. The *intersection* of two fuzzy sets A and B is a fuzzy set C, written as $C = A \cap B$, whose membership function is $f_C(x) = \min\{f_A(x), f_B(x)\}$ for all $x \in X$, or in abbreviated form $f_C(x) = f_A(x) \wedge f_B(x)$. Two fuzzy sets A and B are *disjoint* if $A \cap B$ is empty, i.e., $f_A(x) \wedge f_B(x) = 0$ for all $x \in X$ [6].

Definition 2.2 (Support of a fuzzy set) [2], [4] *The* support *of a fuzzy set A within the universal set X is the set* $\operatorname{supp}(A) = \{x \in X \mid f_A(x) > 0\}$

The support of a fuzzy set A is the set $\operatorname{supp}(A)$ that contains all the elements in X that have nonzero membership grades in A. In the following, we suppose that X is the real Euclidean space $\mathbb{R}^n$, with $n \in \mathbb{N}$.

Definition 2.3 (Convex set) [3], [5] *A nonempty subset A of $\mathbb{R}^n$ is* convex *if for all $x_1, x_2 \in A$ and all $\lambda \in [0,1]$, the point $\lambda x_1 + (1-\lambda)x_2$ belongs to A*

Definition 2.4 (Convex fuzzy set) [6] *A fuzzy set A is* convex *if and only if the sets $\Gamma_\alpha(A)$ defined by $\Gamma_\alpha(A) = \{x \in X \mid f_A(x) \geq \alpha\}$ are convex for all α in the interval* $(0,1]$

3 Partition of a nonempty fuzzy set in nonempty convex fuzzy subsets

Definition 3.1 *A* partition in nonempty convex fuzzy subsets *of a nonempty fuzzy set A is a collection of fuzzy sets*

$$\mathcal{A} = \{A_i \mid i \in I, A_i \subseteq A, A_i \neq \emptyset\},$$

where I is an index set, such that

1. *if $i \neq j$ then $A_i \cap A_j = \emptyset$,*
2. *$\bigcup_{i \in I} A_i = A$,*
3. *for all $i \in I$, A_i is a convex fuzzy set.*

Theorem 3.2 *Every nonempty fuzzy set has at least one partition in nonempty convex fuzzy subsets.*

Proof. We will prove this theorem with the construction of a trivial partition in nonempty convex fuzzy subsets. Let A be a nonempty fuzzy set with membership function $f_A : X \mapsto [0, 1]$, then, $\mathrm{supp}(A) \neq \emptyset$ because A is nonempty, i.e., there exists at least one $x \in X$ such that $f_A(x) > 0$. Now, let us define for every $y \in \mathrm{supp}(A)$ the fuzzy subset $A_y \subseteq A$ with membership function $f_{A_y} : X \mapsto [0, 1]$ which is given by

$$f_{A_y}(x) = \begin{cases} f_A(x) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $A_y \neq \emptyset$ because $f_{A_y}(x) = f_A(x) > 0$ if $x = y \in \mathrm{supp}(A)$. Then, $\mathcal{A} = \{A_y \mid y \in \mathrm{supp}(A)\}$ is a partition in nonempty convex fuzzy subsets of the nonempty fuzzy set A. Now we will prove this fact.

1. If $y_1 \neq y_2$ then $A_{y_1} \cap A_{y_2} = \emptyset$. This is because
   (a) if $x = y_1$ then $f_{A_{y_1}}(x) \wedge f_{A_{y_2}}(x) = f_A(x) \wedge 0 = 0$
   (b) if $x = y_2$ then $f_{A_{y_1}}(x) \wedge f_{A_{y_2}}(x) = 0 \wedge f_A(x) = 0$
   (c) if $x \neq y_1$ and $x \neq y_2$ then $f_{A_{y_1}}(x) \wedge f_{A_{y_2}}(x) = 0 \wedge 0 = 0$
2. Let

$$B = \bigcup_{y \in \mathrm{supp}(A)} A_y$$

We have to prove that $B = A$, or equivalently, $f_B(x) = f_A(x)$ for all $x \in X$. But $f_B(x) = \sup\left\{f_{A_y}(x) \mid y \in \mathrm{supp}(A)\right\}$ for all $x \in X$, then,

(a) if $x \notin \operatorname{supp}(A)$ then

$$f_B(x) = \sup\left\{ f_{A_y}(x) \mid y \in \operatorname{supp}(A) \right\} = \sup\{0\} = 0 = f_A(x)$$

(b) if $x \in \operatorname{supp}(A)$ then

$$f_B(x) = \sup\left\{ f_{A_y}(x) \mid y \in \operatorname{supp}(A) \right\} = \sup\{0, f_A(x)\} = f_A(x)$$

3. For all $y \in \operatorname{supp}(A)$, A_y is a convex fuzzy set. This is because $\Gamma_\alpha(A_y) = \left\{ x \in X \mid f_{A_y}(x) \geq \alpha \right\} = \{y\}$ for all α in the interval $(0, 1]$, then $y \in \Gamma_\alpha(A_y)$ (y is the only member in $\Gamma_\alpha(A_y)$) implies that $\lambda y + (1 - \lambda) y = y \in \Gamma_\alpha(A_y)$ for all $\lambda \in [0, 1]$.

This completes the proof. ■

Figure 1 shows the concept of partition of a nonempty fuzzy set A in three nonempty disjoint convex fuzzy subsets A_1, A_2 and A_3. A fuzzy set can be partitioned in many ways, then, this kind of partition is not unique.

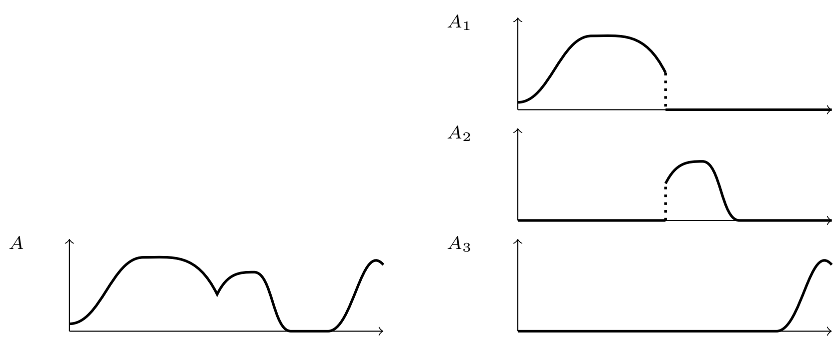

Figure 1: Example of partition of a nonempty fuzzy set in nonempty convex fuzzy subsets

4 Conclusion

This paper presented the concept of partition of a nonempty fuzzy set in nonempty convex subsets. It showed that a nonempty fuzzy set has a least one partition of this kind. A nonempty fuzzy set can be partitioned in many ways, where every member of the partition is a nonempty convex fuzzy subset.

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