Applied Mathematical Sciences, Vol. 6, 2012, no. 58, 2881 - 2889

Electronic Solution of Boolean Equations

William Aristizabal Botero

Department of Physics, Universidad de Caldas
wiarbo@yahoo.es

Alvaro Salas

Universidad de Caldas
Universidad Nacional de Colombia
FIZMAKO Research Group
asalash202@yahoo.com

Abstract

In this paper we show the way a given boolean equation may be solved with the aid of an electronic device.

Keywords: Boolean algebra, boolean function, boolean polynomial, boolean equation

1 Introduction

We begin by defining the concept of a boolean function.

Definition 1.1 *A Boolean function of n variables is a function f of $\mathcal{B}^n$ into $\mathcal{B}$, where $\mathcal{B}$ is the set $\{0,1\}$, n is a positive integer, and $\mathcal{B}^n$ denotes the n-fold cartesian product of the set $\mathcal{B}$ with itself. A point $X^* = (x_1, x_2, ..., x_n) \in \mathcal{B}^n$ is a true point (resp. false point) of the Boolean function f if $f(X^*) = 1$ (resp. $f(X^*) = 0$). We denote by $T(f)$ (resp. $F(f)$) the set of true points (resp. false points) of f. We denote by $\mathbf{1}_n$ the function which takes constant value 1 on $\mathcal{B}^n$ and by $\mathbf{0}_n$ the function which takes constant value 0 on $\mathcal{B}^n$.*

We adhere to the convention that $\mathcal{B} = \{0, 1\}$, where 0 and 1 can be viewed either as abstract symbols or as numerical quantities. The most elementary way to define a Boolean function f is to provide its truth table, i.e. to give a complete list of all the points in $\mathcal{B}^n$ together with the value of the function at each point. Boolean functions can be described in many alternative ways. In this paper, we concentrate on a type of representation derived from propositional logic, namely the representation of Boolean functions by Boolean polynomials.
Our definition of Boolean polynomials will be inductive, starting with three elementary operations as building blocks.

Definition 1.2 *The binary operations $+$ (disjunction, Boolean or) and $\cdot$ (conjunction, Boolean and) and the unary operation $'$ (complementation, negation, Boolean not) are defined on $\mathcal{B}$ as follows:*

$$0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 1;$$

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1;$$

$$1' = 0, \quad 0' = 1.$$

We can naturally extend the definitions of all three elementary operations to $\mathcal{B}^n$ by writing, for all $X = (x_1, ..., x_n) \in \mathcal{B}^n$, $Y = (y_1, ..., y_n) \in \mathcal{B}^n$,

$$X + Y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n),$$

$$X \cdot Y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$\text{and} \quad X' = (x_1', x_2', ..., x_n').$$

Usually, we write $x \cdot y = xy$.

Let us enumerate some of the elementary properties of disjunction, conjunction and complementation.

Theorem 1.1 *For all $x, y, z \in \mathcal{B}$, the following identities hold:*

1. $x + 1 = 1$ *and* $x \cdot 0 = 0$.

2. $x + 0 = x$ *and* $x \cdot 1 = 1 \cdot x = x$.

3. $x + y = y + x$ *and* $xy = yx$ *(Commutativity).*

4. $(x + y) + z = x + (y + z)$ *and* $x(yz) = (xy)z$ *(Associativity).*

5. $x + x = x$ *and* $xx = x$ *(Idempotency).*

6. $x + (xy) = x$ *and* $x(x + y) = x$ *(Absorption).*

7. $x + (yz) = (x + y)(x + z)$ *and* $x(y + z) = (xy) + (xz)$ *(Distributivity).*

8. $x + x' = 1$ *and* $x \cdot x' = 0$.

9. $x'' = x$ *(Involution).*

10. $(x + y)' = x'y'$ *and* $(xy)' = x' + y'$ *(De Morgan's laws).*

11. $x + (xy) = x + y$ *and* $x(x + y) = xy$.

Building upon Definition 1.1, we are now in a position to introduce the important notion of Boolean polynomial.

Definition 1.3 *Given a finite collection of Boolean variables* $x_1, x_2, \ldots, x_n$*, a Boolean polynomial (or Boolean formula or Boolean expression) in the variables* $x_1, x_2, \ldots, x_n$ *is defined as follows:*
(1) *The constants* 0, 1 *and the variables* $x_1, x_2, \ldots, x_n$ *are Boolean polynomials in* $x_1, x_2, \ldots, x_n$ *;*
(2) *If* ϕ *and* ψ *are Boolean polynomials in* $x_1, x_2, \ldots, x_n$*, then* $\phi + \psi$*,* $\phi \cdot \psi$ *and* ϕ' *are Boolean polynomials in* $x_1, x_2, \ldots, x_n$*;*
(3) *Every Boolean polynomial is formed by finitely many applications of the rules (1)-(2). We also say that a Boolean polynomial in the variables* $x_1, x_2, \ldots, x_n$ *is a Boolean polynomial on* $\mathcal{B}^n$.

We use notations like $\phi(x_1, x_2, ..., x_n)$ or $\psi(x_1, x_2, ..., x_n)$ to denote Boolean expressions in the variables $x_1, x_2, \ldots, x_n$. Some examples of boolean polynomials are : $\phi_1(x) = x$, $\phi_2(x) = x'$, $\psi_1(x, y, z) = (x' + y)(y + z') + (xy)z$ and $\psi_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$.
Now, the relation between Boolean polynomials and Boolean functions deserves to be carefully explained. First, since disjunction, conjunction and

complementation can be viewed as Boolean polynomials, every Boolean polynomial $\phi(x_1, x_2, ..., x_n)$ can similarly be viewed as a Boolean function defined by composition: for every point $(x_1^*, x_2^*, ..., x_n^*) \in \mathcal{B}^n$, the value of $\phi(x_1^*, x_2^*, ..., x^n*)$ is obtained by substituting x_i^* for x_i $(i = 1, 2, ..., n)$ in the expression ϕ and by recursively applying Definition 1.2 to the resulting expression. Thus, for example, if $\psi_1(x, y, z) = (x' + y)(y + z') + (xy)z$, then $\psi_1(0, 0, 0) = (0' + 0)(0 + 0') + (00)0 = 1$.
Viewing Boolean polynomials as functions naturally leads to the following definition.

Definition 1.4 *Let f be a Boolean function defined on $\mathcal{B}^n$ and $\psi = \psi(x_1, x_2, ..., x_n)$ be a Boolean polynomial in the variables $x_1, x_2, \ldots, x_n$. We say that ψ represents f (or that ψ expresses f, or that f admits the representation or the expression ψ) if $f(x_1, x_2, ..., x_n) = \psi(x_1, x_2, ..., x_n)$ for all $(x_1, x_2, ..., x_n) \in \mathcal{B}^n$. When this is the case, we simply write $f = \psi$.*

Thus, for example, The polynomial $\psi_1(x, y, z) = (x' + y)(y + z') + (xy)z$ represents the function f, where:

$$f(0, 0, 1) = f(1, 0, 0) = f(1, 0, 1) = 0;$$

$$f(0, 0, 0) = f(0, 1, 0) = f(0, 1, 1) = f(1, 1, 0) = f(1, 1, 1) = 1.$$

It is important to understand that every Boolean function can be represented by numerous Boolean polynomials or expressions, whereas every Boolean expression represents a unique function. As a matter of fact, there are "only' 2^{2^n} Boolean functions of n variables, but there are infinitely many Boolean expressions in n variables. These remarks motivate the distinction that we draw between functions and polynomials. They also lead to the next definition.

Definition 1.5 *We say that two Boolean polynomials ϕ and ψ are equivalent if they represent the same Boolean function. When this is the case, we write $\phi = \psi$.*

Observe, in particular, that any two expressions which can be deduced from each other by repeated use of the properties listed in Theorem 1.1 are equivalent. For example, The function $f(x, y, z)$ represented by the polynomial

$\psi_1(x,y,z) = (x'+y)(y+z') + (xy)z$ is also represented by the expression $\phi = x'z' + y$. Indeed,

$$\begin{aligned}(x'+y)(y+z') + xyz = &\quad (x'y + x'z' + yy + yz') + xyz && \text{(distributivity)}\\ = &\quad x'y + x'z' + y + yz' + xyz && \text{(idempotency and associativity)}\\ = &\quad x'z' + y && \text{(absorption)}\end{aligned}$$

Thus, $\psi_1(x,y,z)$ and $\phi(x,y,z)$ are equivalent, i.e. $\psi_1(x,y,z) = \phi(x,y,z)$.

Definition 1.6 *Let $f : \mathcal{B}^n \to \mathcal{B}$ and $g : \mathcal{B}^n \to \mathcal{B}$ be Boolean functions in the variables $x_1, x_2, \ldots, x_n$. We say that these functions are equal iff*

$$f(x_1,x_2,\ldots,x_n) = g(x_1,x_2,\ldots,x_n) \quad \forall\ (x_1,x_2,\ldots,x_n) \in \mathcal{B}^n.$$

We introduce a partial ordering in the set $\mathfrak{B}_n$ of all Boolean functions from $\mathcal{B}^n$ to $\mathcal{B}$. We consider $0 < 1$.

Definition 1.7 *Let $f : \mathcal{B}^n \to \mathcal{B}$ and $g : \mathcal{B}^n \to \mathcal{B}$ be boolean functions in the variables $x_1, x_2, \ldots, x_n$. We say that f is less or equal g and we write $f \leq g$ iff*

$$f(x_1,x_2,\ldots,x_n) \leq g(x_1,x_2,\ldots,x_n) \quad \forall\ (x_1,x_2,\ldots,x_n) \in \mathcal{B}^n.$$

It is easy to see that $f = g$ iff $f \leq g$ and $g \leq f$. The relation $\leq$ on $\mathfrak{B}_n$ is a partial ordering.
Let ϕ and ψ be Boolean polynomials in the variables $x_1, x_2, \ldots, x_n$. We write $\phi \leq \psi$ iff $f \leq g$, where f and g are the Boolean functions they represent.

Theorem 1.2 *Let ϕ and ψ be Boolean polynomials in the variables $x_1, x_2, \ldots, x_n$. The following conditions are equivalent :*

1. $\phi \leq \psi$.
2. $\phi + \psi = \psi$.
3. $\phi \cdot \psi = \phi$.
4. $\phi \cdot \psi' = 0_n$.
5. $\psi + \phi' = 1_n$.

2 Boolean equations

Like algebraic equations, we may consider Boolean ones.

Definition 2.8 *A Boolean equation is an equation of the form*

$$\phi(x_1, x_2, \ldots, x_n) = \psi(x_1, x_2, \ldots, x_n), \tag{2.1}$$

where ϕ *and* ψ *are Boolean polynomials in the variables* x_1, x_2, . . . , x_n.
A solution of the Boolean equation (2.1) is an element $X^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \mathcal{B}^n$ *for which the equality*

$$\phi(x_1^*, x_2^*, \ldots, x_n^*) = \psi(x_1^*, x_2^*, \ldots, x_n^*),$$

holds. The solution set of (2.1) is the set of all solutions of this equation.

Example 2.1 *Let us consider the equation* $x_1 + x_2x_3 = (x_1 + x_2)x_3$. *Its solutions are* $X_1^* = (0, 0, 0)$, $X_2^* = (0, 0, 1)$, $X_3^* = (0, 1, 0)$, $X_4^* = (0, 1, 1)$, $X_5^* = (1, 0, 1)$ *and* $X_6^* = (1, 1, 1)$.

Another type of Boolean equation is an equation with one or more unknowns.

Definition 2.9 *Let* ϕ *and* ψ *be Boolean polynomials in the variables* x_1, x_2, . . . , x_n, y_1, y_2, ..., y_m. *A Boolean equation in the unknowns* y_1, y_2, ..., y_m *with respect to* x_1, x_2, ..., x_n *is an equation*

$$\phi(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = \psi(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m), \tag{2.2}$$

where x_1,x_2,..., x_n *are arbitrary elements of* $\mathcal{B}$.
A solution of the equation (2.2) is a collection of Boolean polynomials

$$y_1 = \varphi_1(x_1, x_2, \ldots, x_n),\ y_2 = \varphi_2(x_1, x_2, \ldots, x_n),\ y_m = \varphi_m(x_1, x_2, \ldots, x_n)$$

such that the equality (2.2) holds for all $(x_1, x_2, \ldots, x_n) \in \mathcal{B}^n$.

Example 2.2 *Some solutions of the equation* $x_1x_2 + y_1 = x_1 + x_2$ *in the unknown* y_1 *are*

$$y_1^{(1)} = \varphi_1^{(1)}(x_1, x_2) = x_1x_2' + x_1'x_2 \quad \textit{and} \quad y_1^{(2)} = \varphi_1^{(2)}(x_1, x_2) = x_1 + x_2.$$

We also may consider systems of boolean equations.

Definition 2.10 *Let $f(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n)$ be a boolean equation. The direct solution of this equation is the set of n-tuples $(x_1, x_2, \dots, x_n)$ for which both the left side term $f(x_1, x_2, \dots, x_n)$ and the right side one $g(x_1, x_2, \dots, x_n)$ take the value 1. The complemented solution is the set of all n-tuples such that both terms take the value 0.*
The direct and complemented solutions are denoted by $S_{\mathcal{D}}$ and $S_{\mathcal{C}}$, respectively. The general solution S is the union of these solutions, that is $S = S_{\mathcal{D}} \cup S_{\mathcal{C}}$.

3 Electronic solution of boolean equations

The electronic solution of a boolean equation is obtained when each one of the terms of this equation is associated with a logical circuit built with logical gates. Later, both circuits are connected through a XNOR gate and all combinations of input values of the logical variables are introduced. The n-tuples that are solutions of the equation produce an output equal to 1. The general solution is composed by the union of two parts : the direct and the complemented solutions.

Example 3.3 *Find the electronic solution of equation in Example 2.2. See Figure 1.*

The output of this circuit is equal to 1 when both outputs corresponding to the logical circuits for left and right terms are equal either to 0 or 1. When introducing all the possible combinations of the values in the input variables, it is found that the ordered triples that give for result 1 are

$$\{(0,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}.$$

Of these, only the input $(0,0,0)$ is a part of the complemented solution; the other ones correspond to the direct solution.

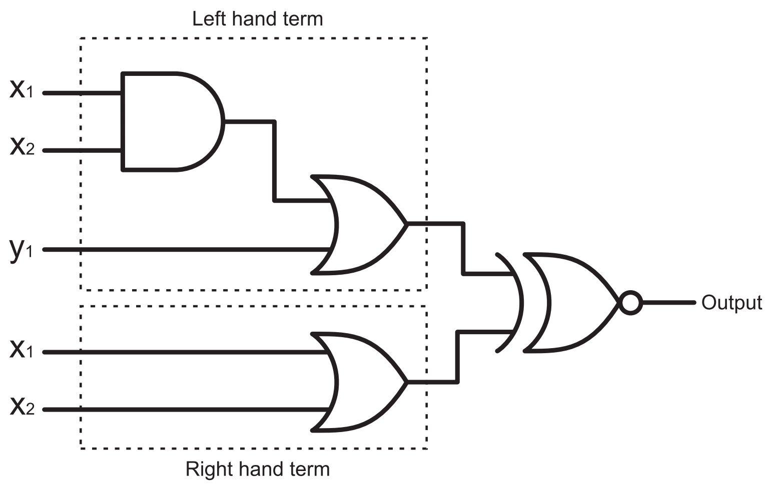

Figure 1: Circuit for the equation $x_1x_2 + y_1 = x_1 + x_2$

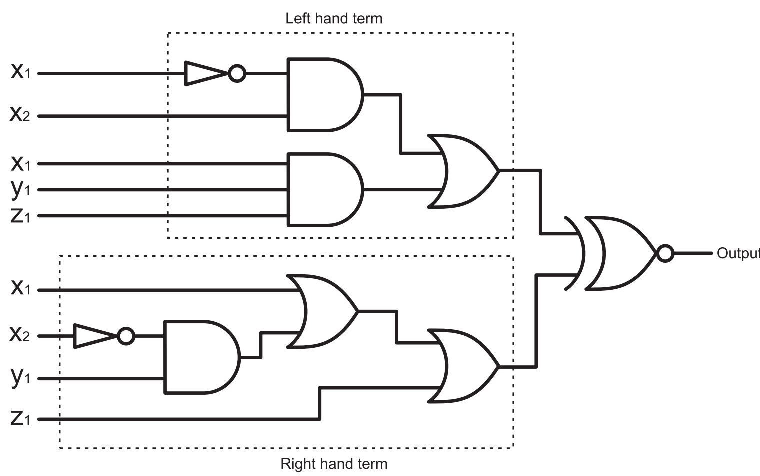

Figure 2: Circuit for the equation $x_1 + x_2x_3 = (x_1 + x_2)x_3$

4 Conclusions

We may apply boolean equations to solve some problems relating teaching of logic.

References

[1] JOHN HOOLEY, *Boole's Method for Solving Boolean Equations*, The Mathematical Gazette, Vol. 50, No. 372. (May, 1966), pp. 114-118.

[2] RALPH M. TOMS, *Systems of Boolean Equations*, The American Mathematical Monthly, Vol. 73, No. 1. (Jan., 1966), pp. 29-35.

[3] DRAGI´C BANKOVI´C, *Boolean inequations*, Discrete Mathematics, **307** (2007) pp. 750 – 755.

[4] SERGIU RUDEANU, *On equations in bounded lattices*, An. St. Univ.- Ovidius Constanta, Vol 1(**9**),2001, pp. 101-106.

Received: January, 2012