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Solving Fuzzy Duffing's Equation by the Laplace Transform Decomposition

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Abstract

The Laplace transform decomposition algorithm (LTDA) is a numerical algorithm which can be adapted to solve the Fuzzy Duffing equations (FDE). This paper will describe the principle of LTDA and discusses its advantages. Concrete examples are also studied to show the numerical results on how LTDA efficiently work to solve the Duffing equations in the fuzzy setting.

Keywords: Laplace transform decomposition algorithm, differential equations, Fuzzy Duffing equations, numerical algorithm, Maple.

1 Introduction

Various problems involving nonlinear control systems in physics, engineering and communication theory, makes increasing interest to study the periodically forced Duffing's equation [6, 10-12]. In many cases, the Duffing's equations have their uncertainty known as fuzzy Duffing's equation (FDE). The Adomian decomposition method (ADM) can be used to find the solutions of a large class of nonlinear problems [1, 5, 7, 8, 13, 14]. However, the implementation of the ADM mainly depends upon the calculation of Adomian polynomials for the nonlinear operators and their numerical solutions were obtained in the form of finite series.

The present paper aims at offering an alternative method to get solution of the FDE by using the Laplace transform numerical scheme on the basis of the decomposition method [12] named Laplace transform decomposition method (LTDM), with fuzzy initial value problems. In this case, in order to solve the FDE, we have applied the Laplace. The decomposition method has been introduced by Adomian [9] to solve linear and nonlinear functional equations such as in algebraic, differential, partial differential, integral, etc. The Adomian decomposition method has been proved to be reliable, accurate and effective in both the analytic and the numerical purposes [2, 7]. In this study, the initial values in the form of fuzzy numbers will be transformed in the parametric forms. Fuzzy numbers are considered because many situations in real-world are facing uncertainty and vagueness [3, 4].

The decomposition method consists of splitting the given equation into linear and nonlinear terms. The linear term $y(x)$ represents an infinite sum of components $y_n(x), n = 0,1,2, \ldots$ defined by

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

The decomposition method identifies the nonlinear term $F(y(x))$ by the decomposition series

$$F(y(x)) = \sum_{n=0}^{\infty} A_n$$

where A_ns are Adomian's polynomial of $y_0, y_1, y_2, \ldots, y_n$. An iterative algorithm is achieved for the determination of $y_n s$ in recursive manner. By using the Maple software, the truncated sum is calculated.

$$y(x) = \sum_{n=0}^{K} y_n$$

This paper is organized as follows. In section 2, we bring some basic definitions of fuzzy numbers and how the LTDA is applied in solving FDE. In section 3, the algorithm is practically applied in relation to the numerical examples, and the numerical results show that the LTDA approximates the exact solution with a high degree of accuracy using only few terms of the iterative scheme. In section 4, we discuss the conclusion and future research.

2 Preliminaries

This section will begin with defining the notation used in this paper. We place a ~ sign over a letter to denote a fuzzy subset of the real numbers. We write $\tilde{A}(x)$, a number in [0, 1] for the membership function of $\tilde{A}$ evaluates at x. An α-cut of $\tilde{A}$, written by $\tilde{A}(\alpha)$,is defined as $\{x|\tilde{A} \geq \alpha\}$, for $0 < \alpha \leq 1$ and α–cuts of fuzzy numbers are always closed and bounded. We represent an arbitrary fuzzy number by an ordered pair of functions $\left(\underline{u}(\alpha), \overline{u}(\alpha)\right)$, which satisfies the following requirements:

1. $\underline{u}(\alpha)$ is a bounded left continuous non decreasing function over [0,1],
2. $\overline{u}(\alpha)$ is a bounded left continuous non increasing function over [0,1],
3. $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$.

A crisp number α is simply represented by $\underline{u}(\alpha) = \overline{u}((\alpha) = \alpha$, $0 \leq \alpha \leq 1$.
For arbitrary fuzzy numbers $x = (\underline{x}(\alpha), \overline{x}(\alpha))$, $y = (\underline{y}(\alpha), \overline{y}(\alpha))$ and real number k.

1. $x = y$ if and only if $\underline{x}(\alpha) = \underline{y}(\alpha)$ and $\overline{x}(\alpha) = \overline{y}(\alpha)$,
2. $x + y = (\underline{x}(\alpha) + \underline{y}(\alpha), \bar{x}(\alpha) + \overline{y}(\alpha))$,
3. $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Let F be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level sets. If $\propto = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$, then the parametric forms for $\propto$ and β are $(\underline{\propto}, \overline{\propto}) = (a_1 + (a_2 - a_1)r, a_3 + (a_2 - a_3)r)$ and $(\underline{\beta}, \overline{\beta}) = (b_1 + (b_2 - b_1)r, b_3 + (b_2 - b_3)r)$ respectively.
The Laplace transform Decomposition Method is applied to find the solution to the following nonlinear fuzzy initial value problems:

$$y'' + py' + p_1 y + p_2 y^3 = f(x), \tag{2.1}$$

$$y(0) = (\underline{\propto}, \overline{\propto}) \text{ and } y'(0) = (\underline{\beta}, \overline{\beta}), \tag{2.2}$$

where p, p_1, and p_2 are real constants and $\propto$ and β in parametric forms.
The method consists of applying Laplace transformation (denoted throughout this paper by

$$\mathcal{L}[y''] + \mathcal{L}[py'] + \mathcal{L}[p_1 y] + \mathcal{L}[p_2 y^3] = \mathcal{L}[f(x)]. \tag{2.3}$$

By using linearity of Laplace transform, we obtain

$$\mathcal{L}[y''] + p\mathcal{L}[y'] + p_1\mathcal{L}[y] + p_2\mathcal{L}[y^3] = \mathcal{L}[f(x)] \tag{2.4}$$

Applying the formulas on Laplace transform, we obtain

$$s^2\mathcal{L}[y] - y(0)s - y'(0) + p\{s\mathcal{L}[y] - y(0)\} + p_1\mathcal{L}[y] + p_2\mathcal{L}[y^3] = \mathcal{L}[f(x)]. \tag{2.5}$$

using the initial condition (2.2), we have

$$(s^2 + ps)\mathcal{L}[y] = (\underline{\propto}, \overline{\propto})s + (\underline{\beta}, \overline{\beta}) + (\underline{\propto}, \overline{\propto})p - p_1\mathcal{L}[y] - p_2\mathcal{L}[y^3] + (\mathcal{L}[f(x)] \tag{2.6}$$

or

$$\mathcal{L}[y] = \frac{(\underline{\propto}, \overline{\propto})s + (\underline{\beta}, \overline{\beta}) + (\underline{\propto}, \overline{\propto})p}{s^2 + ps} - \frac{p_1}{s^2 + ps}\mathcal{L}[y] - \frac{p_2}{s^2 + ps}\mathcal{L}[y^3] + \frac{1}{s^2 + ps}\mathcal{L}[f(x)]. \quad (2.7)$$

In other words Eq. (2.7) are in the fuzzy forms written as lower and upper cases.

$$\mathcal{L}\left[\underline{y}\right] = \frac{\underline{\propto}\, s + \underline{\beta} + \underline{\propto}\, p}{s^2 + ps} - \frac{p_1}{s^2 + ps}\mathcal{L}[y] - \frac{p_2}{s^2 + ps}\mathcal{L}[y^3] + \frac{1}{s^2 + ps}\mathcal{L}[f(x)]. \quad (2.8)$$

$$\mathcal{L}[\overline{y}] = \frac{\overline{\propto}\, s + \overline{\beta} + \overline{\propto}\, p}{s^2 + ps} - \frac{p_1}{s^2 + ps}\mathcal{L}[y] - \frac{p_2}{s^2 + ps}\mathcal{L}[y^3] + \frac{1}{s^2 + ps}\mathcal{L}[f(x)]. \quad (2.9)$$

The Laplace transform decomposition technique consists next of representing the solution as an infinite series, in particular

$$y = \sum_{n=0}^{\infty} y_n, \quad (2.10)$$

where the term y_n are to recursively calculated. Also the nonlinear operator $h(y) = y^3$ is decomposed as

$$h(y) = y^3 = \sum_{n=0}^{\infty} A_n, \quad (2.11)$$

where the A_ns are Adomian polynomials of $y_0, y_1, \dots, y_n$ and are calculated by the formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[h\sum_{i=0}^{\infty}\lambda^i y_i\right]\right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (2.12)$$

The first few polynomials are given by

$$A_0 = h(y_0),$$
$$A_1 = y_1 h^1(y_0),$$
$$A_2 = y_2 h^1(y_0) + \frac{1}{2}{y_1}^2 h^2(y_0),$$
$$A_3 = y_3 h^1(y_0) + \frac{1}{2}y_1 y_2 h^2(y_0) + \frac{1}{6}{y_1}^3 h^3(y_0), \quad (2.13)$$

and for $h(y) = y^3$ they are given by

$$A_0 = {y_0}^3,$$
$$A_1 = 3y_0 y_1,$$

$$A_2 = 3{y_0}^2 y_2 + 3y_0 {y_1}^2, \tag{2.14}$$
$$A_3 = 3y_0 y_3 + 6y_0 y_1 y_2 + {y_1}^3,$$

Substituting (2.10), and (2.11) to (2.8) and (2.9) respectively, the results are:

$$\mathcal{L}\left[\sum_{n=0}^{\infty} \underline{y}\right] = \frac{\underline{\propto}\, s + \underline{\beta} + \underline{\propto}\, p}{s^2 + ps} - \frac{p_1}{s^2 + ps}\mathcal{L}\left[\sum_{n=0}^{\infty} \underline{y}\right] - \frac{p_2}{s^2 + ps}\mathcal{L}\left[\sum_{n=0}^{\infty} \underline{A}_n\right] + \frac{1}{s^2 + ps}\mathcal{L}[f(x)] \tag{2.15}$$

$$\mathcal{L}\left[\sum_{n=0}^{\infty} \overline{y}\right] = \frac{\overline{\propto}\, s + \overline{\beta} + \overline{\propto}\, p}{s^2 + ps} - \frac{p_1}{s^2 + ps}\mathcal{L}\left[\sum_{n=0}^{\infty} \overline{y}\right] - \frac{p_2}{s^2 + ps}\mathcal{L}\left[\sum_{n=0}^{\infty} \overline{A}_n\right] + \frac{1}{s^2 + ps}\mathcal{L}[f(x)]. \tag{2.16}$$

Matching both sides of (2.15) and (2.16), the following iterative algorithms are obtained.

For lower case:

$$\mathcal{L}\left[\underline{y_0}\right] = \frac{\underline{\propto}\, s + \underline{\beta} + \underline{\propto}\, p}{s^2 + ps} + \frac{1}{s^2 + ps}\mathcal{L}[f(x)] \tag{2.17}$$

$$\mathcal{L}\left[\underline{y_1}\right] = -\frac{p_1}{s^2 + ps}\mathcal{L}\left[\underline{y_0}\right] - \frac{p_2}{s^2 + ps}\mathcal{L}\left[\underline{A_0}\right], \tag{2.18}$$

$$\mathcal{L}\left[\underline{y_2}\right] = -\frac{p_1}{s^2 + ps}\mathcal{L}\left[\underline{y_1}\right] - \frac{p_2}{s^2 + ps}\mathcal{L}\left[\underline{A_1}\right], \tag{2.19}$$

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$$\mathcal{L}\left[\underline{y_{n+1}}\right] = -\frac{p_1}{s^2 + ps}\mathcal{L}\left[\underline{y_n}\right] - \frac{p_2}{s^2 + ps}\mathcal{L}\left[\underline{A_n}\right]. \tag{2.20}$$

For upper case:

$$\mathcal{L}\left[\overline{y}_0\right] = \frac{\overline{\propto}\, s + \overline{\beta} + \overline{\propto}\, p}{s^2 + ps} + \frac{1}{s^2 + ps}\mathcal{L}[f(x)] \tag{2.21}$$

$$\mathcal{L}\left[\overline{y}_1\right] = -\frac{p_1}{s^2 + ps}\mathcal{L}\left[\overline{y}_0\right] - \frac{p_2}{s^2 + ps}\mathcal{L}\left[\overline{A}_0\right], \tag{2.22}$$

$$\mathcal{L}\left[\overline{y}_2\right] = -\frac{p_1}{s^2 + ps}\mathcal{L}\left[\overline{y}_1\right] - \frac{p_2}{s^2 + ps}\mathcal{L}\left[\overline{A}_1\right], \tag{2.23}$$

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$$\mathcal{L}[\overline{y}_{n+1}] = -\frac{p_1}{s^2+ps}\mathcal{L}[\overline{y}_n] - \frac{p_2}{s^2+ps}\mathcal{L}[\overline{A}_n]. \quad (2.24)$$

Applying the inverse Laplace transform to (2.17) and (2.21) we obtain the value of $(\underline{y}_0, \overline{y}_0)$. Substituting these values of y_0 to (2.18) and (2.22) respectively, we evaluating the Laplace transform of the quantities on the right side of $\mathcal{L}[\underline{y}_1, \overline{y}_2]$, and then applying the inverse Laplace transform, we obtain the values of $(\underline{y}_1, \overline{y}_2)$. The other terms $y_2, y_3, \ldots$ in lower and upper cases can be obtained recursively in a similar way by using (2.20) and (2.24) respectively. Since the complicated excitation term $f(x)$ can cause difficult integrations and proliferation of terms, we can express $f(x)$ in Taylor series at $x_0 = 0$, which is truncated for simplification. If we replace $f(x)$ by

$$\tilde{f}(x) = \sum_{i=0}^{k} a_i x^i, \quad a_i = \frac{f^{(i)}(0)}{i!}, \quad i = 0,1,2,\ldots,k. \quad (2.25)$$

Eq. (2.17) and (2.21) become

$$\mathcal{L}[\underline{y}_0] = \frac{\underline{\propto}\, s + \underline{\beta} + \underline{\propto}\, p}{s^2+ps} + \frac{1}{s^2+ps}\sum_{i=0}^{k}\frac{a_i i!}{s^{i+1}} \quad (2.26)$$

$$\mathcal{L}[\overline{y}_0] = \frac{\overline{\propto}\, s + \overline{\beta} + \overline{\propto}\, p}{s^2+ps} + \frac{1}{s^2+ps}\sum_{i=0}^{k}\frac{a_i i!}{s^{i+1}} \quad (2.27)$$

By using LTDA which is described above, we obtain values $\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \ldots$

3 Numerical example

The Laplace transform decomposition algorithm is illustrated by the following example [8].

Example. Consider the Duffing's equation in the following type:

$$y'' + 3y - 2y^3 = \cos x \sin 2x \quad (3.1)$$

with initial conditions

$$\tilde{y}(0) = (-1,0,1) \quad \tilde{y}'(0) = (0,1,2) \quad (3.2)$$

The analytic solution of this equation is

$$y(x) = \sin(x) \quad (3.3)$$

By using ADM, this example was considered in [11] and its solution (3.3) was found. However, in this study our initial value problem will be written in the parametric forms of the fuzzy setting and the equation is then solved by LTDM.

If we replace

$$\tilde{f}(x) = 2x - \frac{7}{3}x^3 + \frac{61}{60}x^5 - \frac{547}{2520}x^7. \tag{3.4}$$

Eq. (3.1) becomes

$$y'' + 3y - 2y^3 = \tilde{f}(x) \tag{3.5}$$

with fuzzy initial condition (3.2).

If we reapply the given above algorithm, considering $p = 0, p_1 = 3, p_2 = -2$, $\propto = (-1,0,1), \beta = (0,1,2)$ to Eq. (3.5), we obtain following iterative algorithm:

$$\mathcal{L}[\tilde{y}_0] = \frac{1}{s^2} + \frac{1}{s^2}\mathcal{L}[\tilde{f}], \tag{3.6}$$

$$\mathcal{L}[\tilde{y}_1] = -\frac{3}{s^2}\mathcal{L}[\tilde{y}_0] + \frac{2}{s^2}\mathcal{L}[\tilde{A}_0], \tag{3.7}$$

$$\mathcal{L}[\tilde{y}_2] = -\frac{3}{s^2}\mathcal{L}[\tilde{y}_1] + \frac{2}{s^2}\mathcal{L}[\tilde{A}_1], \tag{3.8}$$

in general,

$$\mathcal{L}[\tilde{y}_{n+1}] = -\frac{3}{s^2}\mathcal{L}[\tilde{y}_n] + \frac{2}{s^2}\mathcal{L}[\tilde{A}_n]. \tag{3.9}$$

Substituting the inverse Laplace transform to (3.6) for the *r* =0.8 for the lower case, we obtain

$$\underline{y}_0 = -0.2 + 0.8x + \frac{1}{3}x^3 - \frac{7}{60}x^5 + \frac{61}{2520}x^7 - \frac{547}{181440}x^9 \tag{3.10}$$

Substituting this value of $\underline{y}_0$ and $\underline{A}_0 = \underline{y}_0^3$ given in (2.14) to (3.7), then the result is

$$\mathcal{L}\left[\underline{y}_1\right] = \frac{0.584}{s^3} - \frac{2.208}{s^4} - \frac{1.536}{s^5} + \frac{0.624}{s^6} - \frac{15.36}{s^7} + \cdots \tag{3.11}$$

Operating with Laplace inverse on both sides of (3.7) we get

$$\underline{y}_1 = 0.292x^2 - 0.368x^3 - 0.064x^4 + 0.0052x^5 + \cdots \tag{3.12}$$

Substituting (3.12) to (3.8) and using A_1 given in (2.14), we obtain

$$\mathcal{L}[\underline{y}_2] = -\frac{1.612}{s^5} + \frac{2.730}{s^6} + \frac{48.108}{s^7} - \frac{184.583}{s^8} + \frac{742.994}{s^9} + \cdots \tag{3.13}$$

The inverse Laplace transform applied to (3.13) yields

$$\underline{y}_2 = 0.0228x^5 + 0.0668x^6 - 0.03662x^7 + 0.0184x^8 + \cdots \tag{3.14}$$

Higher iterations can be easily obtained by using the Maple software. For example

$$\underline{y}_3 = 0.0028x^6 + 0.0175x^7 - 0.0292x^8 + 0.0097x^9 + \cdots \tag{3.15}$$

$$\underline{y}_4 = 0.0016x^8 - 0.0070x^9 + 0.0058x^{10} + 0.0022x^{11} + \cdots \tag{3.16}$$

By using Maple, it is easier to get the determinate partial sum $\underline{\phi}_n(x) = \sum_{m=0}^{n} y_m$. In this case, particularly the sum of five terms is calculated as below:

$$\underline{\phi}_5(r = 0.8) = -0.2 + 0.8x + 0.2920x^2 - 0.0347x^3 - 0.1312x^4 + \cdots \quad (3.17)$$

Substituting $x = \frac{\pi}{2} = 1.570796327$ in Eq. (3.17), we have the sum of five terms for $r = 0.8$ is 0.84425**.**

By using the same method in iterations for the upper case, we have series of $\overline{y}_0$, $\overline{y}_1$, $\overline{y}_2$, ... as below:

$$\overline{y}_0 = 0.2 + 1.2x + \frac{1}{3}x^3 - \frac{7}{60}x^5 + \frac{61}{2520}x^7 - \frac{547}{181440}x^9 \quad (3.18)$$

$$\overline{y}_1 = -0.292x^2 - 0.552x^3 + 0.144x^4 + 0.127x^5 + \cdots \quad (3.19)$$

$$\overline{y}_2 = 0.067x^4 + 0.034x^5 - 0.150x^6 - 0.118x^7 + \cdots \quad (3.20)$$

$$\overline{y}_3 = -0.0028x^6 + 0.0262x^7 + 0.0657x^8 + 0.0239x^9 + \cdots \quad (3.20)$$

$$\overline{y}_4 = 0.0001x^8 - 0.0011x^9 - 0.0013x^{10} + 0.0038x^{11} + \cdots \quad (3.21)$$

Also calculated the sum of five terms:

$$\overline{\phi}_5(r = 0.8) = 0.2 + 1.2x - 0.2920x^2 - 0.2187x^3 + 0.2112x^4 + \cdots \quad (3.22)$$

Substituting $x = \frac{\pi}{2} = 1.570796327$ in Eq. (3.22), we have the sum of five terms for $r = 0.8$ is 1.80253.

Meanwhile for $r = 1$, the results obtained are as below:

$$\tilde{y}_0 = x + \frac{1}{3}x^3 - \frac{7}{60}x^5 + \frac{61}{2520}x^7 - \frac{547}{181440}x^9 \quad (3.23)$$

$$\tilde{y}_1 = -\frac{1}{2}x^3 + \frac{1}{20}x^5 + \frac{47}{840}x^7 - \frac{89}{60480}x^9 - \cdots \quad (3.24)$$

$$\tilde{y}_2 = \frac{3}{40}x^5 - \frac{3}{40}x^7 - \frac{523}{20160}x^9 + \frac{18281}{2217600}x^{11} - \cdots \quad (3.25)$$

$$\tilde{y}_3 = -\frac{3}{560}x^7 + \frac{29}{960}x^9 + \frac{859}{739200}x^{11} - \frac{92901}{12812800}x^{13} + \cdots \quad (3.26)$$

$$\tilde{y}_4 = \frac{1}{4480}x^9 - \frac{1843}{246400}x^{11} + \frac{20287}{5491200}x^{13} + \frac{23386949}{8072064000}x^{15} + \cdots \quad (3.27)$$

Also calculated the sum of five terms for

$$\overline{\phi}_5(r = 1.0) = -\frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \cdots \quad (3.28)$$

Substituting $x = \frac{\pi}{2} = 1.570796327$ in Eq. (3.28), we have the sum of five terms for $r = 1.0$ is 1.0000**.**

4 Conclusion

In this paper, investigation on solving FDE by LTDM has been done and exists. In the real-world problems, we always deal with ambiguities condition for example the uncertainties and vagueness in the values of the initial conditions. Therefore, LTDM approach will be taken into consideration as an alternative method to overcome this situation. In solving fuzzy Duffing's equation, we conclude that the meaningful starting point is at r = 0.8. By referring to the result found in this study, we can get better approximation for this type of equations faster.

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