Applied Mathematical Sciences, Vol. 6, 2012, no. 60, 2989 - 3002

Numerical Solution of Fuzzy Differential Equations by Runge Kutta Method of Order Five

T. Jayakumar, D. Maheskumar and K. Kanagarajan

Department of Mathematics
Sri Ramakrishna Mission Vidyalaya College of Arts and science
Coimbatore-641 020, Tamilnadu, India
jayakumar.thippan68@gmail.com, madhana2010@gmail.com

Abstract

In this paper numerical algorithms for solving 'fuzzy ordinary differential equations' based on Seikkala derivative of fuzzy process [9], are considered. A numerical method based on the Runge-Kutta method of order five in detail is discussed and this is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear Fuzzy Cauchy Problems.

Keywords: Fuzzy Differential Equation, Runge-Kutta Method of Order Five, Fuzzy Cauchy Problem

1 Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [3] . It was followed up by D. Dubois, H. Prade in [4] , who defined and used the extension principle. The fuzzy differential equation and the initial value problem were regularly treated by O. Kelva in [7, 8] and by S. Seikkala in [9]. The numerical method for solving fuzzy differential equations is introduced by M.Ma, M.Friedman, A. Kandel in [12] by the standard Euler Method and by authors in [1, 2] by Taylor method.

The structure of this Chapter organizes as follows. In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative, which have been discussed by S. Seikkala in [9], are given. In section 3 we define the problem, this is a fuzzy Cauchy problem whose numerical solution is the main interest of this chapter. The numerically solving fuzzy differential equation by the Runge-Kutta method of order 5 is discussed in section 4. The proposed algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

2 Preliminary Notes

Consider the initial value problem

$$\begin{cases} y^{'}(t) = f(t, y(t)); \quad a \leq t \leq b, \\ \\ y(a) = \alpha, \end{cases} \tag{1}$$

The basis of all Runge-Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i, \tag{2}$$

where for $i = 1, 2, \ldots, m$, the $w_i^{\prime}s$ are constants and

$$k_i = h \cdot f(t_n + \alpha_i h,\ y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j). \tag{3}$$

Equation (2) is to be exact for powers of h through h^m, because it is to be coincident with Taylor series of order m. Therefore, the truncation error T_m, can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}).$$

The true magnitude of γ_m will generally be much less than the bound of theorem 2.1 Thus, if the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1}$, as we expect, to be so if h is small, then the bound on $\gamma_m h^{m+1}$, will usually be a bound on the error as a whole. The famous nonzero constants α_i, β_{ij} in the Runge Kutta method of order 5 are

$\alpha_1 = 0,\ \alpha_2 = \alpha_3 = \frac{1}{3}, \alpha_4 = \frac{1}{2}, \alpha_5 = 1, \beta_{21} = \frac{1}{3},\ \beta_{31} = \beta_{32} = \frac{1}{6}, \beta_{41} = \frac{1}{8},$ $\beta_{42} = 0, \beta_{43} = \frac{3}{8},\ \beta_{51} = \frac{1}{2},\ \beta_{52} = 0, \beta_{53} = \frac{-3}{2},\ \beta_{54} = 2,$

where $m = 5$. Hence we have,

$$
\begin{aligned}
&y_0 = \alpha,\\
&k_1 = h.f(t_i, y_i),\\
&k_2 = h.f(t_i + \tfrac{h}{3}, y_i + \tfrac{k_1}{3}),\\
&k_3 = h.f(t_i + \tfrac{h}{3}, y_i + \tfrac{k_1}{6} + \tfrac{k_2}{6}),\\
&k_4 = h.f(t_i + \tfrac{h}{2}, y_i + \tfrac{k_1}{8} + \tfrac{3k_3}{8}),\\
&k_5 = h.f(t_i + h, y_i + \tfrac{k_1}{2} - \tfrac{3k_3}{2} + 2k_4),\\
&y_{i+1} = y_i + \tfrac{1}{6}(k_1 + 4k_4 + k_5),
\end{aligned}
\tag{4}
$$

where

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{5}$$

Theorem 2.1 *Let $f(t, y)$ belong to $C^4[a, b]$ and let it's partial derivatives are bounded and assume there exists, L, M, positive numbers, such that*

$$|f(t,y)| < M, \quad |\frac{\partial^{i+j} f}{\partial t^i \partial y^j}| < \frac{L^{i+j}}{M^{j-1}}, \quad i + j \leq m,$$

then in the Runge-Kutta method of order 5, $y(t_{i+1}) - y_{i+1} \approx \dfrac{11987}{12960} h^6 M L^5 + O(h^7)$

A triangular fuzzy number v, is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $v(x)$, the membership function of the fuzzy number v, is a triangle with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$. We specify v as $(a_1/a_2/a_3)$. We will write: (1)$v > 0$ if $a_1 > 0$; (2)$v \geq 0$ if $a_1 \geq 0$; (3)$v < 0$ if $a_3 < 0$; and (4)$v \leq 0$ if $a_3 \leq 0$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r-$level intervals. It means that if $v \in E$ then the $r-$level set

$$[v]_r = \{s | \ v(s) \geq r\}, \quad 0 < r \leq 1,$$

is a closed bounded interval which is denoted by

$$[v]_r = [v_1(r) \ , v_2(r)].$$

Let I be a real interval. A mapping $x : I \to E$ is called a fuzzy process and its $r-$level set is denoted by

$$[x(t)]_r = [x_1(t;r)\ , x_2(t;r)], \quad t \in I, \quad r \in (0,1].$$

The derivative $x^{'}(t)$ of a fuzzy process $x(t)$ is defined by

$$[x^{'}(t)]_r = [x_1^{'}(t;r)\ , x_2^{'}(t;r)], \quad t \in I, \quad r \in (0,1],$$

provided that this equation defines a fuzzy number, as in Seikkala [9].

Lemma 2.2 *Let $v, w \in E$ and s scalar, then for $r \in (0,1]$*

$$\Big[v+w\Big]_r = \Big[v_1(r)+w_1(r)\ , v_2(r)+w_2(r)\Big],$$
$$\Big[v-w\Big]_r = \Big[v_1(r)-w_1(r)\ , v_2(r)-w_2(r)\Big],$$
$$\Big[v.w\Big]_r = \Big[min\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\},$$
$$\max\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\}\Big],$$
$$\Big[sv\Big]_r = s\Big[v\Big]_r.$$

3 A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$\begin{cases} y^{'}(t) = f(t, y(t)), \quad t \in I = [0,T], \\ \\ y(0) = y_0, \end{cases} \tag{6}$$

where f is a continuous mapping from $R_+ \times R$ into R and $y_0 \in E$ with $r-$level sets

$$[y_0]_r = [y_1(0;r)\ , y_2(0;r)], \quad r \in (0,1].$$

The extension principle of Zadeh leads to the following definitioin of $f(t, y)$ when $y = y(t)$ is a fuzzy number

$$f(t,y)(s) = sup\Big\{y(\tau)|\ s = f(t,\tau)\Big\}, \quad s \in R.$$

It follows that

$$\Big[f(t,y)\Big]_r = \Big[f_1(t,y;r), f_2(t,y;r)\Big], \quad r \in (0,1],$$

where

$$\begin{aligned} f_1(t,y;r) &= min\Big\{f(t,u)|\ u \in [y_1(r), y_2(r)]\Big\}, \\ \\ f_2(t,y;r) &= max\Big\{f(t,u)|\ u \in [y_1(r), y_2(r)]\Big\}. \end{aligned} \tag{7}$$

Theorem 3.1 *Let f satisfy*

$$|f(t,v) - f(t,\bar{v})| \leq g(t, |v-\bar{v}|), \quad t \geq 0, \quad v, \bar{v} \in R,$$

where $g : R_+ \times R_+$ is a continuous mapping such that $r \rightarrow g(t,r)$ is nondecreasing, the initial value problem

$$u^{'}(t) = g(t, u(t)), \quad u(0) = u_0, \tag{8}$$

has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (8) for $u_0 = 0$. Then the fuzzy initial value problem (6) has a unique fuzzy solution.

4 The Runge-Kutta Method of Order Five

Let the exact solution $[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]$ is approximated by some $[y(t)]_r = [y_1(t;r), y_2(t;r)]$. From (2),(3) we define

$$\begin{aligned} y_1(t_{n+1};r) - y_1(t_n;r) &= \textstyle\sum_{i=1}^{5} w_i k_{i,1}(t_n, y(t_n;r)), \\ y_2(t_{n+1};r) - y_2(t_n;r) &= \textstyle\sum_{i=1}^{5} w_i k_{i,2}(t_n, y(t_n;r)). \end{aligned} \tag{9}$$

where the $w_i' s$ are constants and

$$\begin{aligned} &\Big[k_i(t, y(t;r))\Big]_r = \Big[k_{i,1}\Big(t, y(t,r)), k_{i,2}(t, y(t,r)\Big)\Big], \quad i = 1,2,3,4,5 \\ &k_{i,1}\Big(t, y(t,r)\Big) = h.f\Big(t_n + \alpha_i h, y_1(t_n) + \textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,1}(t_n, y(t_n;r))\Big), \\ &k_{i,2}\Big(t, y(t,r)\Big) = h.f\Big(t_n + \alpha_i h, y_2(t_n) + \textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,2}(t_n, y(t_n;r))\Big), \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
k_{1,1}\Big(t, y(t;r)\Big) &= min\Big\{h.f(t,u)|\ u \in [y_1(t;r), y_2(t;r)]\Big\},\\
k_{1,2}\Big(t, y(t;r)\Big) &= max\Big\{h.f(t,u)|\ u \in [y_1(t;r), y_2(t;r)]\Big\},\\
k_{2,1}\Big(t, y(t;r)\Big) &= min\Big\{h.f(t+\tfrac{h}{3},u)|\ u \in \Big[z_{1,1}\Big(t,y(t;r)\Big), z_{1,2}\Big(t,y(t;r)\Big)\Big]\Big\},\\
k_{2,2}\Big(t, y(t;r)\Big) &= max\Big\{h.f(t+\tfrac{h}{3},u)|\ u \in \Big[z_{1,1}\Big(t,y(t;r)\Big), z_{1,2}\Big(t,y(t;r)\Big)\Big]\Big\},\\
k_{3,1}\Big(t, y(t;r)\Big) &= min\Big\{h.f(t+\tfrac{h}{3},u)|\ u \in \Big[z_{2,1}\Big(t,y(t;r)\Big), z_{2,2}\Big(t,y(t;r)\Big)\Big]\Big\},\\
k_{3,2}\Big(t, y(t;r)\Big) &= max\Big\{h.f(t+\tfrac{h}{3},u)|\ u \in \Big[z_{2,1}\Big(t,y(t;r)\Big), z_{2,2}\Big(t,y(t;r)\Big)\Big]\Big\},\\
k_{4,1}\Big(t, y(t;r)\Big) &= min\Big\{h.f(t+\tfrac{h}{2},u)|\ u \in \Big[z_{3,1}\Big(t,y(t;r)\Big), z_{3,2}\Big(t,y(t;r)\Big)\Big]\Big\},\\
k_{4,2}\Big(t, y(t;r)\Big) &= max\Big\{h.f(t+\tfrac{h}{2},u)|\ u \in \Big[z_{3,1}\Big(t,y(t;r)\Big), z_{3,2}\Big(t,y(t;r)\Big)\Big]\Big\},\\
k_{5,1}\Big(t, y(t;r)\Big) &= min\Big\{h.f(t+h,u)|\ u \in \Big[z_{4,1}\Big(t,y(t;r)\Big), z_{4,2}\Big(t,y(t;r)\Big)\Big]\Big\},\\
k_{5,2}\Big(t, y(t;r)\Big) &= max\Big\{h.f(t+h,u)|\ u \in \Big[z_{4,1}\Big(t,y(t;r)\Big), z_{4,2}\Big(t,y(t;r)\Big)\Big]\Big\}.
\end{aligned} \tag{11}
$$

where in the Runge-Kutta method of order 5,

$$
\begin{aligned}
z_{1,1}\Big(t, y(t;r)\Big) &= y_1(t;r) + \tfrac{1}{3}k_{1,1}\Big(t,y(t;r)\Big),\\
z_{1,2}\Big(t, y(t;r)\Big) &= y_2(t;r) + \tfrac{1}{3}k_{1,2}\Big(t,y(t;r)\Big),\\
z_{2,1}\Big(t, y(t;r)\Big) &= y_1(t;r) + \tfrac{1}{6}k_{1,1}\Big(t,y(t;r)\Big) + \tfrac{1}{6}k_{2,1}\Big(t,y(t;r)\Big),\\
z_{2,2}\Big(t, y(t;r)\Big) &= y_2(t;r) + \tfrac{1}{6}k_{1,2}\Big(t,y(t;r)\Big) + \tfrac{1}{6}k_{2,2}\Big(t,y(t;r)\Big),\\
z_{3,1}\Big(t, y(t;r)\Big) &= y_1(t;r) + \tfrac{1}{8}k_{1,1}\Big(t,y(t;r)\Big) + \tfrac{3}{8}k_{3,1}\Big(t,y(t;r)\Big),\\
z_{3,2}\Big(t, y(t;r)\Big) &= y_2(t;r) + \tfrac{1}{8}k_{1,2}\Big(t,y(t;r)\Big) + \tfrac{3}{8}k_{3,2}\Big(t,y(t;r)\Big),\\
z_{4,1}\Big(t, y(t;r)\Big) &= y_1(t;r) + \tfrac{1}{2}k_{1,1}\Big(t,y(t;r)\Big) - \tfrac{3}{2}k_{3,1}\Big(t,y(t;r)\Big) + 2k_{4,1}\Big(t,y(t;r)\Big),\\
z_{4,2}\Big(t, y(t;r)\Big) &= y_2(t;r) + \tfrac{1}{2}k_{1,2}\Big(t,y(t;r)\Big) - \tfrac{3}{2}k_{3,2}\Big(t,y(t;r)\Big) + 2k_{4,2}\Big(t,y(t;r)\Big).
\end{aligned} \tag{12}
$$

Define,

$$F\Big[t, y(t;r)\Big] = k_{1,1}\Big(t, y(t;r)\Big) + 4k_{4,1}\Big(t, y(t;r)\Big) + k_{5,1}\Big(t, y(t;r)\Big)$$
$$G\Big[t, y(t;r)\Big] = k_{1,2}\Big(t, y(t;r)\Big) + 4k_{4,2}\Big(t, y(t;r)\Big) + k_{5,2}\Big(t, y(t;r)\Big). \tag{13}$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $\Big[Y(t_n)\Big]_r = \Big[Y_1(t_n;r),\ Y_2(t_n;r)\Big]$ and $\Big[y(t_n)\Big]_r = \Big[y_1(t_n;r),\ y_2(t_n;r)\Big]$, respectively. The solution is calculated by grid points at (5). By (9),(13) we have

$$\begin{aligned} Y_1(t_{n+1};r) &\approx\ Y_1(t_n;r) + \tfrac{1}{6}F\Big[t_n, Y(t_n;r)\Big], \\ Y_2(t_{n+1};r) &\approx\ Y_2(t_n;r) + \tfrac{1}{6}G\Big[t_n, Y(t_n;r)\Big]. \end{aligned} \tag{14}$$

We define

$$\begin{aligned} y_1(t_{n+1};r) &=\ y_1(t_n;r) + \tfrac{1}{6}F\Big[t_n, y(t_n;r)\Big], \\ y_2(t_{n+1};r) &=\ y_2(t_n;r) + \tfrac{1}{6}G\Big[t_n, y(t_n;r)\Big]. \end{aligned} \tag{15}$$

The following lemmas will be applied to show convergence of these approximates
i.e.,

$$\begin{aligned} \lim_{h\to 0} y_1(t;r) = Y_1(t;r), \\ \lim_{h\to 0} y_2(t;r) = Y_2(t;r). \end{aligned}$$

Lemma 4.1 *Let the sequence of numbers* $\{W_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N-1,$$

for some given positive constants A *and* B. *Then*

$$|W_n| \leq A^n|W_0| + B\frac{A^n - 1}{A-1}, \quad 0 \leq n \leq N.$$

Lemma 4.2 *Let the sequence of numbers* $\{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N$ *satisfy*

$$\begin{aligned} |W_{n+1}| &\leq |W_n| + A.max\{|W_n|, |V_n|\} + B, \\ |V_{n+1}| &\leq |V_n| + A.max\{|W_n|, |V_n|\} + B. \end{aligned}$$

for some given positive constants A *and* B, *and denote*

$$U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N.$$

Then

$$|Un| \leq \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A} - 1}, \quad 0 \leq n \leq N,$$

where $\overline{A} = 1 + 2A$ *and* $\overline{B} = 2B$.

Let $F(t,u,v)$ *and* $G(t,u,v)$ *are obtained by substituting* $[y(t)]_r = [u,v]$ *in (13),*

$$F(t,u,v) = k_{1,1}(t,u,v) + 4k_{4,1}(t,u,v) + k_{5,1}(t,u,v)$$
$$G(t,u,v) = k_{1,2}(t,u,v) + 4k_{4,2}(t,u,v) + k_{5,2}(t,u,v).$$

The domain where F *and* G *are defined is therefore*
$K = \Big\{(t,u,v)|\ 0 \leq t \leq T, \quad -\infty < v < \infty, \quad -\infty < u < v\Big\}.$

Theorem 4.3 *Let* $F(t,u,v)$ *and* $G(t,u,v)$ *belong to* $C^4(K)$ *and let the partial derivatives of* F *and* G *be bounded over* K*. Then, for arbitrary fixed* $r, 0 \leq r \leq 1$*, the approximately solutions (14) converge to the exact solutions* $Y_1(t;r)$ *and* $Y_2(t;r)$ *uniformly in* t.

Proof:

It is sufficient to show

$$\lim_{h\to 0} y_1(t_N;r) = Y_1(t_N;r),$$
$$\lim_{h\to 0} y_2(t_N;r) = Y_2(t_N;r),$$

where $t_N = T$*. For* $n = 0,1,\ldots,N-1$*, by using Taylor theorem we get*

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + \tfrac{1}{6}F[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \frac{11987}{12960}h^6 ML^5 + O(h^7),\\ Y_2(t_{n+1};r) &= Y_2(t_n;r) + \tfrac{1}{6}G[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \frac{11987}{12960}h^6 ML^5 + O(h^7), \end{aligned} \quad (16)$$

denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$
$$V_n = Y_2(t_n;r) - y_2(t_n;r).$$

Hence from (15) and (16)

$$W_{n+1} = W_n + \tfrac{1}{6}\Big\{F[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\} + \frac{11987}{12960}h^6 ML^5 + O(h^7),$$

$$V_{n+1} = V_n + \tfrac{1}{6}\Big\{G[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\} + \frac{11987}{12960}h^6 ML^5 + O(h^7).$$

Then

$$|W_{n+1}| \leq |W_n| + \tfrac{1}{3}Lh.max\Big\{|W_n|,|V_n|\Big\} + \frac{11987}{12960}h^6ML^5 + O(h^7),$$

$$|V_{n+1}| \leq |V_n| + \tfrac{1}{3}Lh.max\Big\{|W_n|,|V_n|\Big\} + \frac{11987}{12960}h^6ML^5 + O(h^7),$$

for $t \in [0,T]$ *and* $L > 0$ *is a bound for the partial derivatives of* F *and* G*. Thus by lemma 4.2*

$$|W_n| \leq (1+\frac{2}{3}Lh)^n|U_0| + \left(\frac{11987}{6480}h^6ML^5 + O(h^7)\right)\frac{(1+\frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

$$|V_n| \leq (1+\frac{2}{3}Lh)^n|U_0| + \left(\frac{11987}{6480}h^6ML^5 + O(h^7)\right)\frac{(1+\frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

where $|U_0| = |W_0| + |V_0|$*. In particular*

$$|W_n| \leq (1+\frac{2}{3}Lh)^n|U_0| + \left(\frac{11987}{4320}h^5ML^5 + O(h^6)\right)\frac{(1+\frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L},$$

$$|V_n| \leq (1+\frac{2}{3}Lh)^n|U_0| + \left(\frac{11987}{4320}h^6ML^5 + O(h^7)\right)\frac{(1+\frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L}.$$

Since $W_0 = V_0 = 0$*, we obtain*

$$|W_n| \leq \left(\frac{11987}{4320}ML^4\right)\frac{e^{\frac{2}{3}LT} - 1}{L}h^4 + O(h^6),$$

$$|V_n| \leq \left(\frac{11987}{4320}ML^4\right)\frac{e^{\frac{2}{3}LT} - 1}{L}h^4 + O(h^6),$$

and if $h \to 0$ *we get* $W_N \to 0$ *and* $V_N \to 0$ *which completes the proof.*

5 NUMERICAL EXAMPLES

Example 5.1 *Consider the fuzzy initial value problem, [12],*

$$\begin{cases} y'(t) = y(t), \quad t \in I = [0,1], \\ y(0) = (.75 + .25r,\ 1.125 - .125r), \quad 0 < r \leq 1. \end{cases}$$

By using the Runge-Kutta method of order 5, we have

$$y_1(t_{n+1};r) = y_1(t_n;r)\Big[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{144}\Big],$$
$$y_2(t_{n+1};r) = y_2(t_n;r)\Big[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{144}\Big].$$

The exact solution is given by

$$Y_1(t;r) = y_1(0;r)e^t, \quad Y_2(t;r) = y_2(0;r)e^t,$$

which at $t = 1$,

$$Y_1(1;r) = \Big[(.75 + .25r)e, \quad (1.125 - .125r)e\Big], \quad 0 < r \leq 1.$$

The exact and approximate solutions by Improved Euler mathod and the Runge Kutta method of order 5, are compared and plotted at $t = 1$ *in figure 1.*

Table 1

r	*Improved Euler's Method*		*Runge Kutta Method of order 5*		*Exact Solution*	
	$y_1(t_i;r)$	$y_2(t_i;r)$	$y_1(t_i;r)$	$y_2(t_i;r)$	$Y_1(t_i;r)$	$Y_2(t_i;r)$
0.01	1.9812	2.9705	2.0453	3.0544	2.0394	3.0578
0.1	2.0465	2.9377	2.1064	3.0237	2.1067	3.0241
0.2	2.1125	2.9047	2.1743	2.9897	2.1746	2.9901
0.3	2.1785	2.8717	2.2423	2.9558	2.2426	2.9561
0.4	2.2446	2.8387	2.3102	2.9216	2.3105	2.9222
0.5	2.3106	2.8057	2.3781	2.8877	2.3785	2.8882
0.6	2.3766	2.7727	2.4460	2.8537	2.4465	2.8542
0.7	2.4425	2.7396	2.5141	2.8198	2.5144	2.8202
0.8	2.5087	2.7066	2.5820	2.7858	2.5824	2.7862
0.9	2.5746	2.6736	2.6500	2.7518	2.6503	2.7523
1	2.6406	2.6406	2.7179	2.7179	2.7183	2.7183

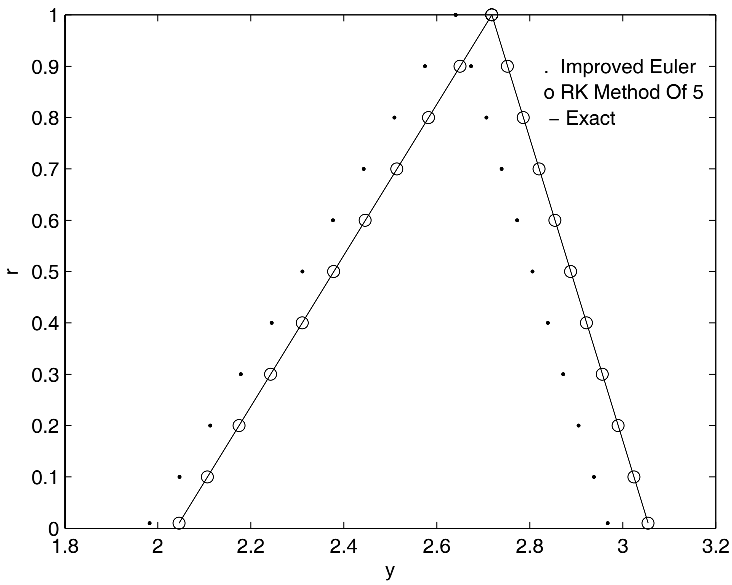

Figure 1: h=0.5

Example 5.2 *Consider the fuzzy initial value problem*

$$y^{'}(t) = c_1 y^2(t) + c_2, \quad y(0) = 0,$$

where $c_i > 0$, *for* $i = 1, 2$ *are triangular fuzzy numbers, [13].*

The exact solution is given by

$$Y_1(t; r) = l_1(r) tan(w_1(r)t),$$
$$Y_2(t; r) = l_2(r) tan(w_2(r)t),$$

with

$$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)}, \quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)},$$
$$w_1(r) = \sqrt{c_{1,1}(r).c_{2,1}(r)}, \quad w_2(r) = \sqrt{c_{1,2}(r).c_{2,2}(r)},$$

where

$$[c_1]_r = [c_{1,1}(r),\ c_{1,2}(r)] \quad and \quad [c_2]_r = [c_{2,1}(r),\ c_{2,2}(r)]$$

$$c_{1,1}(r) = 0.5 + 0.5r, \quad c_{1,2}(r) = 1.5 - 0.5r,$$
$$c_{2,1}(r) = 0.75 + 0.25r, \quad c_{2,2}(r) = 1.25 - 0.25r.$$

The $r-$level sets of $y^{'}(t)$ are

$$Y_1^{'}(t;r) = c_{2,1}(r)sec^2(w_1(r)t),$$
$$Y_2^{'}(t;r) = c_{2,2}(r)sec^2(w_2(r)t),$$

which defines a fuzzy number. We have
$f_1(t,y;r) = min\Big\{c_1.u^2 + c_2 | u \in [y_1(t;r),\ y_2(t;r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)],$
$c_2 \in [c_{2,1}(r),\ c_{2,2}(r)]\Big\},$
$f_2(t,y;r) = max\Big\{c_1.u^2 + c_2 | u \in [y_1(t;r),\ y_2(t;r)], c_1 \in [c_{1,1}(r),\ c_{1,2}(r)],$
$c_2 \in [c_{2,1}(r),\ c_{2,2}(r)]\Big\}.$
By using the Runge-Kutta method of order 5 at $t_n,\ 0 \leq n \leq N$

$$\begin{aligned}
k_{1,1}(t_n;r) &= h(c_{1,1}(r).y_1^2(t_n;r) + c_{2,1}(r)),\\
k_{1,2}(t_n;r) &= h(c_{1,2}(r).y_2^2(t_n;r) + c_{2,2}(r)),\\
k_{2,1}(t_n;r) &= h(c_{1,1}(r).z_{1,1}^2(t_n;r) + c_{2,1}(r)),\\
k_{2,2}(t_n;r) &= h(c_{1,2}(r).z_{1,2}^2(t_n;r) + c_{2,2}(r)),\\
k_{3,1}(t_n;r) &= h(c_{1,1}(r).z_{2,1}^2(t_n;r) + c_{2,1}(r)),\\
k_{3,2}(t_n;r) &= h(c_{1,2}(r).z_{2,2}^2(t_n;r) + c_{2,2}(r)),\\
k_{4,1}(t_n;r) &= h(c_{1,1}(r).z_{3,1}^2(t_n;r) + c_{2,1}(r)),\\
k_{4,2}(t_n;r) &= h(c_{1,2}(r).z_{3,2}^2(t_n;r) + c_{2,2}(r)),\\
k_{5,1}(t_n;r) &= h(c_{1,1}(r).z_{4,1}^2(t_n;r) + c_{2,1}(r)),\\
k_{5,2}(t_n;r) &= h(c_{1,2}(r).z_{4,2}^2(t_n;r) + c_{2,2}(r)),
\end{aligned}$$

where

$$\begin{aligned}
z_{1,1}(t_n;r) &= y_1(t_n;r) + \frac{1}{3}k_{1,1}(t_n;r),\\
z_{1,2}(t_n;r) &= y_2(t_n;r) + \frac{1}{3}k_{1,2}(t_n;r),\\
z_{2,1}(t_n;r) &= y_1(t_n;r) + \frac{1}{6}k_{1,1}(t_n;r) + \frac{1}{6}k_{2,1}(t_n;r),\\
z_{2,2}(t_n;r) &= y_2(t_n;r) + \frac{1}{6}k_{1,2}(t_n;r) + \frac{1}{6}k_{2,2}(t_n;r),\\
z_{3,1}(t_n;r) &= y_1(t_n;r) + \frac{1}{8}k_{1,1}(t_n;r) + \frac{3}{8}k_{3,1}(t_n;r),\\
z_{3,2}(t_n;r) &= y_2(t_n;r) + \frac{1}{8}k_{1,2}(t_n;r) + \frac{3}{8}k_{3,2}(t_n;r),\\
z_{4,1}(t_n;r) &= y_1(t_n;r) + \frac{1}{2}k_{1,1}(t_n;r) - \frac{3}{2}k_{3,1}(t_n;r) + 2k_{4,1}(t_n;r),\\
z_{4,2}(t_n;r) &= y_2(t_n;r) + \frac{1}{2}k_{1,2}(t_n;r) - \frac{3}{2}k_{3,2}(t_n;r) + 2k_{4,2}(t_n;r).
\end{aligned}$$

The exact and approximate solutions are shown in figure 2 at $t = 1$.

Table 2

r	*Improved Euler's Method*		*Runge kutta Method of order 5*		*Exact Solution*	
	$y_1(t_i;r)$	$y_2(t_i;r)$	$y_1(t_i;r)$	$y_2(t_i;r)$	$Y_1(t_i;r)$	$Y_2(t_i;r)$
0.01	0.8727	2.8729	0.8649	3.9406	0.8650	4.3914
0.1	0.9128	2.6962	0.9078	3.5140	0.9079	3.7886
0.2	0.9666	2.5160	0.9584	3.1224	0.9585	3.2851
0.3	1.0205	2.3511	1.0128	2.8014	1.0129	2.8994
0.4	1.0775	2.2001	1.0714	2.5314	1.0715	2.5918
0.5	1.1386	2.0612	1.1344	2.3039	1.1348	2.3419
0.6	1.2036	1.9336	1.2034	2.1096	1.2038	2.1330
0.7	1.2733	1.1860	1.2785	1.9419	1.2793	1.9568
0.8	1.3479	1.7074	1.3610	1.7957	1.3625	1.8051
0.9	1.4278	1.6069	1.4524	1.6674	1.4545	1.6732
1	1.5141	1.5141	1.5537	1.5537	1.5574	1.5574

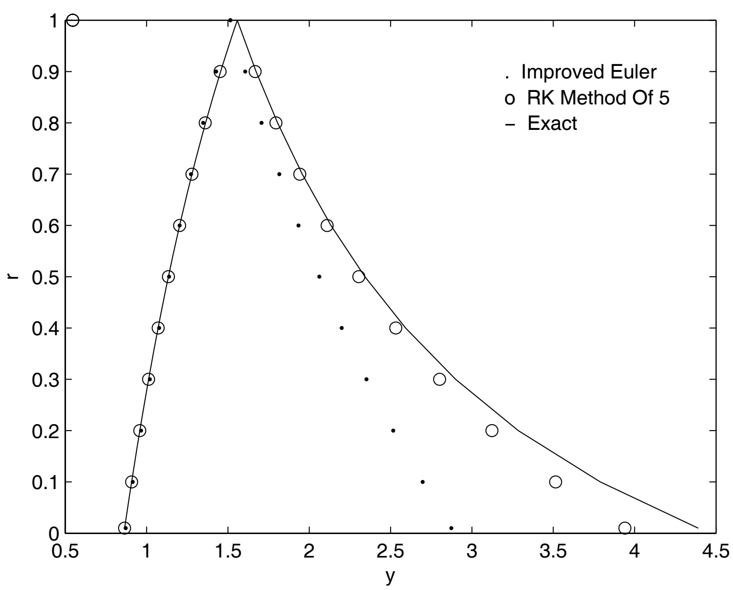

Figure 2: h=0.5

6 Conclusion

In this work we have applied iterative solution of *Runge-kutta Method of order five* for numerical solution of fuzzy differential equations. It is clear that the method introduced in Chapter with $O(h^5)$ performs better than *Improved Euler's Method* with $O(h^2)$.

References

[1] S. Abbasbandy, T. Allah Viranloo, Numerical solution of fuzzy differential equations by Taylor method, Journal of Computational Methods in Applied Mathematics 2(2002), 113-124.

[2] S. Abbasbandy, T. Allah Viranloo, Numerical solution of fuzzy differential equation, Mathematical and Computational Applications 7 (2002),41-52.

[3] S.L. Chang, L.A. Zadeh, On Fuzzy Mapping and Control, IEEEE Trans.*Systems Men Cybernet.* 2 (1972), 30-34.

[4] D. Dubois, H. Prade,Towards Fuzzy Differential Calculus:Part 3, Differentiation,*Fuzzy Sets and System,*8 (1982), 225-233.

[5] M.L Puri, D. A. Ralescu, Differentials of fuzzy functions, *J. Math. Anal.Appl.*91 (1983),321-325.

[6] R. Goetschel, W. Voxman, Elemenary fuzzy calculus, *Fuzzy Sets and Systems*18 (1986),31-43.

[7] O. Kaleva, Fuzzy Differential Equations, *Fuzzy Sets and Systems,* 24 (1987), 301-317.

[8] O. Kaleva, The Cauchy Problem for Fuzzy Differential Equations, *Fuzzy Sets and Systems,* 35 (1990), 389-396.

[9] S. Seikkala, On the Fuzzy Initial value problem, *Fuzzy Sets and Systems,* 24 (1987), 319-330.

[10] R. L. Burden, J. D. Faires, Numerical Analysis, 1997.

[11] A. Ralston, P. Rabinowitz, First Course In Numerical Analysis, 1978.

[12] M. Ma, M. Friedman and A. kandel, Numerical Solutions of Fuzzy Differential Equations, *Fuzzy Sets and Systems,* 105 (1999), 133-138.

[13] James J. Buckley, Thomas Feurihg, Fuzzy Differential Equations, *Fuzzy Sets and Systems,*110 (2000),43-54.

Received: January, 2012