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Dynamic Mean-Variance Portfolio Selection with Liability and No-Shorting Constraints

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Abstract

In this paper, we formulate a mean-variance portfolio selection model with liability under the constraint that short-selling is prohibited. Due to the introduction of the liability and no-shorting constraints, our problem is not a conventional stochastic optimal linear-quadratic(LQ) control problem, and the corresponding HJB equation has no continuous solution. we construct a lower-semicontinuous function through two Ricttati equations, and show that this function is a viscosity solution of the HJB equation. we get explicitly the optimal dynamic strategy and the mean-variance efficient frontier in closed forms.

Keywords: Mean-variance portfolio selection; HJB equation; viscosity supersolution; liability; no short-selling

1 Introduction

The mean-variance (M-V)portfolio selection problem is proposed by Markowitz (1952) for a single period, and is extended to multi-period and continuous-time portfolio selection. The main difficulty of this paper is that the variance is hard to analyze due to its non-separability in the sense of dynamic programming. Recently, Li et al.(2000) and Zhou et al.(2000) studied the embedding technique to overcome this difficulty in the multi-period and continuous-time setting, respectively.

In this paper, we investigate the continuous-time mean-variance portfolio selection problem with the liability in the case where short-selling is not allowed. As a consequence,the methods used by Xun et al.(2002)do not apply directly. Using the idea of embedding, our problem can be converted into a stochastic LQ optimal control problem with constraints. Note that the control is constrained, the Riccatic approach does not apply directly. We solve this problem by studying the HJB equation. It is noteworthy that Xun et al.(2002) construct a continuous function as the viscosity solution to HJB equation. However, due to the introduction of the liability , the solution to the HJB equation derived from our model is not continuous. We overcome this difficulty by constructing a lower-semicontinuous solution to the HJB equation via two Ricttati equations. Last, we prove that this function is a viscosity solution of the HJB equation, and obtain the optimal dynamic strategy and the mean-variance efficient frontier in closed forms using the viscosity solution verification theorem.

2 Model

Let $(\Omega, \mathcal{F}, P, F)$be a complete filtered probability space. Consider an investor equipped with an initial endowment $\omega > 0$ and an initial liability $l(l \in R)$ at time $t = 0$.Denote by x the net initial wealth of the investor ,i.e.,$x = \omega - l$.The investor is allowed to adjust his/her portfolio during the time interval $[0, T]$.

We consider a financial market in which 2 assets are trade continuously within the time horizon $[0, T]$. oth asset is a risk-free asset whose price S_t^0 satisfies the following differential equation

$$\begin{cases} dS_t^0 = r_t S_t^0 dt, \quad t \in [0, T] \\ S_0^0 = 1. \end{cases} \tag{1}$$

$$\begin{cases} dS_t = S_t \mu_t dt + S_t \sigma_t dW(t), \quad t \in [0, T], \\ S_0 = a \in R. \end{cases} \tag{2}$$

Throughout this paper we assume that S_t^0, S_t and σ_t are deterministic and bounded on $[0, T]$.

$$dL_t = u_t dt + v_t dB_t, L_0 = l, \tag{3}$$

where $\{B_t : t \in [0, T]\}$is a one-dimensional standard Brownian motion define on the underlying probability space $(\Omega, \mathcal{F}, P, F)$.

Similarly Xie etal(2008), we denote by ρ_t the correlation coefficient between B_t and W_t. B_t can be expressed as $B_t = \rho_t W_t + \sqrt{1 - \rho_t^2} W_t^0$. (3) can be rewritten as

$$dL_t = u_t dt + v_t \rho_t dW_t + v_t \sqrt{1 - \rho_t^2} dW_t^0, L_0 = l. \tag{4}$$

We denote by X_t the net wealth of the investor at time $t \in [0, T]$, and by π_t the dollar amount invested in risky asset at time t. The process π_t is also called a strategy. It follows from the formula (9) given in [11] that the wealth of the investor satisfies

$$dX_t = (r_t X_t + b_t \pi_t - u_t)dt + (\pi_t \sigma_t + \delta_t)dW_t + \delta_t^0 dW_t^0, X_0 = x. \qquad (5)$$

where $b_t = (\mu_t - r_t 1) > 0, \delta_t = -v_t \rho_t, \delta_t^0 = -v_t \sqrt{1 - \rho_t^2}$. The admissible strategy set with initial wealth x is defined as $\mathcal{A}(x)$.

The M-V portfolio problem and can be expressed as the bi-objective optimization problem:

$$\mathbb{P} \min_{\pi \in \mathcal{A}(x)} (-EX_T, VarX_T) \qquad (6)$$

It is known from Li and Ng(2000) that $\mathbb{P}$ is equivalent to the following single objective optimization problem:

$$\mathbb{P}(\lambda) \min_{\pi \in \mathcal{A}(x)} (-EX_T + \lambda VarX_T), \qquad (7)$$

where the parameter $\lambda \in [0, \infty)$ represents the weight imposed by the investor on the objective $VarX_T$.

Define

$$\Pi_{\mathbb{P}}(\lambda) := \{\pi | \pi \text{ is an optimal strategy of } \mathbb{P}(\lambda)\}. \qquad (8)$$

3 Solution of an Auxiliary Problem

Similar to Zhou and Li(2000),we introduce an auxiliary problem as follows:

$$\mathbb{A}(\lambda, \omega) \quad \min_{\pi \in \mathcal{A}(x)} E(\lambda X_T^2 - \omega X_T), \qquad (9)$$

where $\omega \in \mathbb{R}$is given beforehand.

Define

$$\Pi_{\mathbb{A}(\lambda,\omega)} := \{\pi | \pi \text{ is an optimal strategy of } \mathbb{A}(\lambda, \omega)\}. \qquad (10)$$

Following Zhou and Li(2000),the relationship between problem $\mathbb{P}(\lambda)$ and $\mathbb{A}(\lambda, \omega)$can be shown as below.

Theorem 1 (Zhou and Li(2000)). For any $\lambda > 0$,one has $\prod_{\mathbb{P}(\lambda)} \subseteq \bigcup_{\omega \in \mathbb{R}} \prod_{\mathbb{A}(\lambda,\omega)}$. Moreover,if $\pi^* \in \prod_{\mathbb{P}(\lambda)}$,then$\pi^* \in \prod_{\mathbb{A}(\lambda,\omega)}$ with $\omega^* = 1 + 2\lambda EX_T^*$,where X_t^* is the wealth process corresponding to the strategy π^*.

The optimal cost functional of the problem $\mathbb{A}(\gamma)$ is define as

$$V(t, x) = \inf_{\pi \geq 0} E[\lambda X_T^2 - \omega X_T | \mathcal{F}_t]. \qquad (11)$$

First, we derive the HJB equation of $\mathbb{A}(\lambda,\omega)$, which is the following partial differential equation

$$\begin{cases} \frac{\partial V}{\partial t}(t,x)+\inf_{\pi\geq 0}\{\frac{\partial V}{\partial x}(t,x)[r_t x_t+b_t\pi-u_t+\frac{1}{2}\frac{\partial^2 V}{\partial y^2}(t,x)[((\sigma_t\pi_t)+\delta_t)^2+\delta_t^{0^2}]=0, \\ V(T,X)=\lambda x^2-\omega x, \end{cases} \tag{12}$$

Let

$$V(t,x)=\frac{1}{2}\bar{P}_t x^2+\bar{g}_t x+\bar{c}_t, \tag{13}$$

(14)can be written as

$$\begin{array}{l}(\frac{1}{2}\dot{\bar{P}}_t+\bar{P}_t r_t)x^2+(\dot{\bar{g}}_t-\bar{P}_t u_t+\bar{g}_t r_t)x+(\dot{\bar{c}}_t-\bar{g}_t u_t+\frac{1}{2}\bar{P}_t\delta_t^2+\frac{1}{2}\bar{P}_t\delta_t^{0^2}) \\ +\frac{1}{2}\bar{P}_t\inf_{\pi\geq 0}\{\sigma_t^2\pi_t^2+(2\sigma_t\delta_t+2b_t(x-\gamma)+2\bar{\eta}_t b_t)\pi_t\}=0,\end{array} \tag{14}$$

where $\bar{\eta}_t=\frac{\bar{g}_t}{\bar{P}_t}$, $\gamma:=\frac{\omega}{2\lambda}$.

When $\sigma_t\delta_t+b_t y+b_t\bar{\eta}_t<0$, we have

$$\pi_t^*=-\frac{\sigma_t\delta_t+b_t(x-\gamma)+b_t\bar{\eta}_t}{\sigma_t^2}. \tag{15}$$

Let $\bar{P}(t),\bar{g}(t)$ and $\bar{c}(t)$, respectively, denote the solutions of the following differential equations

$$\begin{cases} \dot{\bar{P}}_t=[-2r_t+\frac{b_t^2}{\sigma_t^2}]\bar{P}_t, \\ \bar{P}_T=2\lambda, \\ \bar{P}_t>0 \quad \forall \ t\in[0,T], \end{cases}
\begin{cases} \dot{\bar{g}}_t=[-r_t+\frac{b_t^2}{\sigma_t^2}]\bar{g}_t+[u_t+\frac{\delta_t b_t}{\sigma_t}]\bar{P}_t, \\ \bar{g}_T=-\omega, \end{cases}
\begin{cases} \dot{\bar{c}}_t=\bar{g}_t\mathcal{K}_t-\frac{1}{2}\bar{P}\delta_t^{0^2}+\frac{1}{2}\frac{b_t^2}{\sigma_t^2}\bar{P}_t^{-1}\bar{g}_t^2, \\ \bar{c}_T=0, \end{cases} \tag{16}$$

where $\mathcal{K}_t=u_t+\frac{\delta_t b_t}{\sigma_t}$.

When $\sigma_t\delta_t+b_t x+b_t\bar{\eta}_t>0$,we have $\pi_t^*=0$, and $\tilde{P}(t),\tilde{g}(t)$ and $\tilde{c}(t)$,respectively,denote the solutions of the following differential equations

$$\begin{cases} \dot{\tilde{P}}_t=-2r_t\tilde{P}_t, \\ \tilde{P}(T)=2\lambda, \\ \tilde{P}(t)>0 \quad \forall \ t\in[0,T], \end{cases}
\begin{cases} \dot{\tilde{g}}_t=-r_t\tilde{g}_t+u_t\tilde{P}_t, \\ \tilde{g}_T=-\omega, \end{cases}
\begin{cases} \dot{\tilde{c}}_t=u_t\tilde{g}_t-\frac{1}{2}\tilde{P}_t\delta_t^{0^2}-\frac{1}{2}\tilde{P}_t{\delta_t}^2, \\ \tilde{c}_T=0. \end{cases} \tag{17}$$

Theorem 2 The optimal value function of the problem $\mathbb{A}(\gamma)$ is

$$V(t,x)=\begin{cases}\frac{1}{2}\bar{P}_t x^2+\bar{g}_t x+\bar{c}_t,\\ \qquad \text{if } x-\gamma+\int_t^T-\mathcal{K}_t e^{\int_s^t r_z dz}dt-\gamma e^{-\int_t^T r_s ds}+\frac{\sigma_t\delta_t}{b_t}\le 0,\\ \frac{1}{2}\tilde{P}_t y^2+\tilde{g}_t y+\tilde{c}_t,\\ \qquad \text{if } x-\gamma+\int_t^T-\mathcal{K}_t e^{\int_s^t r_z dz}dt-\gamma e^{-\int_t^T r_s ds}+\frac{\sigma_t\delta_t}{b_t}>0,\end{cases} \tag{18}$$

which is a viscosity superssolution of the HJB equation (12). The optimal control strategy of $\mathbb{A}(\lambda,\omega)$ is

$$\pi_t^*=\begin{cases}-\frac{b_t x+\delta_t\sigma_t-b_t\gamma+b_t\eta_t}{\sigma_t^2},\\ \qquad \text{if } x-\gamma+\int_t^T-\mathcal{K}_t e^{\int_s^t r_z dz}dt-\gamma e^{-\int_t^T r_s ds}+\frac{\sigma_t\delta_t}{b_t}\le 0,\\ 0 \quad \text{if } x-\gamma+\int_t^T-\mathcal{K}_t e^{\int_s^t r_z dz}dt-\gamma e^{-\int_t^T r_s ds}+\frac{\sigma_t\delta_t}{b_t}>0,\end{cases} \tag{19}$$

Where $\eta_t=\int_t^T-\mathcal{K}_t e^{\int_s^t r_z dz}dt-\gamma e^{-\int_t^T r_s ds}$.

Proof Solving (16) yields $\bar{P}_t=2\lambda e^{\int_t^T(2r_s-\frac{B_t^2}{\sigma_t^2})ds}$ and $\bar{g}_t=(\int_t^T-\mathcal{K}_t e^{\int_s^t r_z dz}dt-\gamma e^{-\int_t^T r_s ds})2\lambda e^{\int_t^T(2r_s-\frac{B_t^2}{\sigma_t^2})ds}$. From (15), (19) can be obtained.

4 Solution of Original Problem

In order to obtain the efficien frontier of original problem, we evaluate $\mathrm{Var}X(T)$.

When $x-\gamma+\int_t^T-\mathcal{K}_t e^{\int_s^t r_z dz}dt-\gamma e^{-\int_t^T r_s ds}+\frac{\sigma_t\delta_t}{b_t}\le 0$, substituting (19) into (4) yields

$$dX_t=[(r_t-\frac{b_t^2}{\sigma_t^2})X_t+\frac{b_t^2}{\sigma_t^2}\gamma-\frac{b_t^2}{\sigma_t^2}\tau_t-\mathcal{K}_t]d_t+[-\frac{b_t}{\sigma_t}(X_t-\gamma+\eta_t)]dW_t+\delta_t^0 dW_t^0. \tag{20}$$

Taking expectation on both sides of (20), we obtain

$$dEX_t=[(r_t-\frac{b_t^2}{\sigma_t^2})EX_t+\frac{b_t^2}{\sigma_t^2}\gamma-\frac{b_t^2}{\sigma_t^2}\eta_t-\mathcal{K}_t]dt. \tag{21}$$

This leads to

$$EX_T=\alpha+\beta\gamma, \tag{22}$$

where $\alpha=-\int_0^T\mathcal{K}_t e^{\int_t^T r_z dz-\int_0^T\frac{b_z^2}{\sigma_z^2}dz}dt+xe^{\int_0^T(r_z-\frac{b_z^2}{\sigma_z^2})dz},\beta=1-e^{-\int_0^T\frac{b_z^2}{\sigma_z^2}dz}$.

Similarly, we have

$$E{X_T}^2=\eta+\beta\gamma^2, \tag{23}$$

where $\eta=2xe^{\int_0^T(r_z-\frac{b_z^2}{\sigma_z^2})dz}\int_0^T-\mathcal{K}_t e^{\int_t^T r_z dz}dt+e^{\int_0^T\frac{b_z^2}{\sigma_z^2}dz}(\int_0^T-\mathcal{K}_t e^{\int_t^T r_z dt-\int_0^T\frac{b_z^2}{\sigma_z^2})dz}d_t)^2+x^2e^{\int_0^T(2r_z-\frac{b_z^2}{\sigma_z^2})dz}+\int_0^T{\delta_t^0}^2 e^{\int_t^T(2r_z-\frac{b_z^2}{\sigma_z^2})dz}dt$.

By Theorem 1 and (22), we obtain $\gamma = \frac{EX_T-\alpha}{\beta}$. Correspondingly, the variance of the terminal wealth is

$$VarX_T = EX_T{}^2 - (EX_T)^2 = \frac{1-\beta}{\beta}[EX_T - \frac{\alpha}{1-\beta}]^2 - \frac{\alpha^2}{1-\beta} + \eta. \qquad (24)$$

When $x - \gamma + \int_t^T -\mathcal{K}_t e^{\int_s^t r_z dz} dt - \gamma e^{-\int_t^T r_s ds} + \frac{\sigma_t \delta_t}{b_t} > 0$, we have

$$EX_T = xe^{\int_0^T r_t dz} - \int_0^T u_s e^{\int_0^t r_t dz} ds,\ VarX_T = \int_0^T (\delta_t^{0\,2} + \delta_t^2) e^{\int_t^T 2r_z dz} dt = \vartheta.$$

Theorem 3 The optimal control strategy of original problem is

$$\pi_t^* = \begin{cases} -\frac{b_t x + \delta_t \sigma_t - b_t \gamma + b_t \eta_t}{\sigma_t^2}, & \\ \qquad \text{if } x - \gamma + \int_t^T -\mathcal{K}_t e^{\int_s^t r_z dz} dt - \gamma e^{-\int_t^T r_s ds} + \frac{\sigma_t \delta_t}{b_t} \le 0, \\ 0 \qquad \text{if } x - \gamma + \int_t^T -\mathcal{K}_t e^{\int_s^t r_z dz} dt - \gamma e^{-\int_t^T r_s ds} + \frac{\sigma_t \delta_t}{b_t} > 0, \end{cases}$$

Where $\gamma^* = \frac{EX_T^* - \alpha}{\beta}$. The efficient frontier is given by

$$VarX_T = \begin{cases} \frac{e^{-\int_0^T \frac{b_z^2}{\sigma_z^2} dz}}{1 - e^{-\int_0^T \frac{b_z^2}{\sigma_z^2} dz}} \times [EX_T^* - (xe^{\int_0^T r_z dz} - \int_0^T \mathcal{K}_t e^{\int_0^T r_z dz} dt)]^2 + \int_0^T \delta_t^{0\,2} e^{\int_t^T (2r_z - \frac{b_z^2}{\sigma_z^2}) dz} dt, \\ \qquad \text{if } x - \gamma + \int_t^T -\mathcal{K}_t e^{\int_s^t r_z dz} dt - \gamma e^{-\int_t^T r_s ds} + \frac{\sigma_t \delta_t}{b_t} \le 0, \\ \vartheta, \qquad \text{if } x - \gamma + \int_t^T -\mathcal{K}_t e^{\int_s^t r_z dz} dt - \gamma e^{-\int_t^T r_s ds} + \frac{\sigma_t \delta_t}{b_t} > 0. \end{cases}$$

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