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Exact Construction of Liu Process

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Abstract

In this paper, we show how to fully mathematically create Liu process in credibility theory.

Keywords: Credibility measure, standard Liu process, Liu process

1 Introduction

Credibility measure was created by Baoding Liu [5, 7] and his Chinese co workers [4] around 2004. Then, Baoding Liu designed a grand view of fuzzy process, hybrid process and uncertain process [6]. In response to his work, Dai [1] sketched how to create standard normal fuzzy process under credibility measure, which he called standard Liu process. Independently Kageyama and Iwamura [3] constructed discrete time credibilistic processes and studied some convergences of the processes. Then, Iwamura and Kageyama [2] showed the existence of fuzzy process called standard Liu process with non decreasing sample paths. In this paper, we show how to mathematically create Liu process. In section 2, we give preliminary facts on credibility measure, fuzzy process, Liu process and credibility extension theorem. In section 3, we state how to create standard Liu process. In section 4, we state how to create Liu process.

2 Preliminary

Let Θ be an arbitrary non-empty set. Let $\mathcal{P}$ be the power set of Θ. We call a real valued set function Cr on $\mathcal{P}$ credibility measure if it satisfies the following four axioms [5, 7].

Axiom 1(Normality). $\text{Cr}\{\Theta\} = 1$.

Axiom 2(Monotonicity). Cr *is increasing, i.e.,* $\text{Cr}\{A\} \le \text{Cr}\{B\}$ *whenever* $A \subset B \in \mathcal{P}$.

Axiom 3(Self-duality). Cr *is self-dual, i.e.,* $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ *for any* $A \in \mathcal{P}$.

Axiom 4(Maximality). $\text{Cr}\{\cup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ *for any* $\{A_i\} \subset \mathcal{P}$ *with* $\sup_i \text{Cr}\{A_i\} < 0.5$.

Each element in $\mathcal{P}$ is called an event. To an event A, a number $\text{Cr}\{A\}$ which indicates the credibility that A will occur is assigned. We call $(\Theta, \mathcal{P}, \text{Cr})$ a credibility space. A fuzzy variable ξ is defined as a function from Θ to the set of real numbers $\Re$. Let $\xi_1, \dots, \xi_n$ be fuzzy variables on Θ. Then $\xi_1, \dots, \xi_n$ are said independent if they satisfy

$$\text{Cr}\{\theta \in \Theta | \xi_1(\theta) \in B_1 \ \& \cdots \& \ \xi_n(\theta) \in B_n\} = \bigwedge_{i=1}^{n} \text{Cr}\{\theta \in \Theta | \xi_i(\theta) \in B_i\}$$

for any subsets $B_1, \dots, B_n$ of $\Re$, where $a \wedge b = \min(a, b)$. A fuzzy process $X(t, \theta)$ is defined as a function from $[0, \infty) \times \Theta$ to $\Re$ such that $X(t^*, \theta)$ is a fuzzy variable for each $t^* \in [0, \infty)$. We write $X_t(\theta)$ or X_t to denote $X(t, \theta)$. Liu process is a type of fuzzy process which was introduced by Liu [6] in 2008.

Definition 2.1 *([6]) A fuzzy process* C_t *is said to be a Liu process if*

(i) $C_0 = 0$,

(ii) C_t *has stationary and independent increments,*

(iii) Every increments $C_{t+s} - C_s$ *is a normally distributed fuzzy variable with expected value* et *and variance* $\sigma^2 t^2$ *whose membership function is*

$$\mu_{e,\sigma}(x) = 2\Big(1 + \exp\Big(\frac{\pi |x - et|}{\sqrt{6}\sigma t}\Big)\Big)^{-1}, \quad -\infty < x < \infty.$$

The parameters e and σ are called the drift and diffusion coefficient, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$. Finally we add two theorems on credibility measure.

Theorem 2.1 *(Liu [5, 7]) Let* $(\Theta, \mathcal{P}, \text{Cr})$ *be a credibility space. Then we have*

$$\left.\begin{array}{l} \sup\limits_{\theta \in \Theta} \text{Cr}\{\theta\} \ge 0.5, \\ \text{Cr}\{\theta^*\} + \sup\limits_{\theta \ne \theta^*} \text{Cr}\{\theta\} = 1 \ \ if \ \ \text{Cr}\{\theta^*\} \ge 0.5. \end{array}\right\} \quad (CEC) \qquad (2.1)$$

We call (CEC) the credibility extension condition.

Theorem 2.2 *(Li and Liu [4], Liu [5], Credibility Extension Theorem) Suppose that* Θ *is a nonempty set, and* $\mathrm{Cr}\{\theta\}$ *is a nonnegative function on* Θ *satisfying the (CEC). Then* $\mathrm{Cr}\{\theta\}$ *has a unique extension to a credibility measure as follows.*

$$\mathrm{Cr}\{A\} = \begin{cases} \sup_{\theta\in A} \mathrm{Cr}\{\theta\} & if \ \sup_{\theta\in A} \mathrm{Cr}\{\theta\} < 0.5, \\ 1 - \sup_{\theta\in A^c} \mathrm{Cr}\{\theta\} & if \ \sup_{\theta\in A} \mathrm{Cr}\{\theta\} \geq 0.5. \end{cases} \tag{2.2}$$

3 Standard Liu Process

Let H be a set of functions θ from $[0,\infty)$ to $\Re^+ = [0,\infty)$ such that

(i) $\theta(0) = 0$,

(ii) $\theta(t)$ is Lipschitz continuous, i.e., θ has the Lipschitz constant $K(\theta)$ as $K(\theta) = \sup_{0\leq s<t} \frac{|\theta(t)-\theta(s)|}{t-s}$.

Then, assign to each element $\theta \in H$ a number $\mathrm{Cr}\{\theta\}$ such that

$$\mathrm{Cr}\{\theta\} = \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} K(\theta)\Big)\Big)^{-1}. \tag{3.1}$$

We verify that Cr satisfies the (CEC) in Theorem 2.1 as follows. Let $\theta_0 \in H$ be such that $\theta_0(t) = 0 (\forall t \geq 0)$. Then $K(\theta_0) = 0$ with

$$\sup_{\theta\in H} \mathrm{Cr}\{\theta\} \geq \mathrm{Cr}\{\theta_0\} = 0.5. \tag{3.2}$$

Next, for $\theta^* \in H$ suppose that $\mathrm{Cr}\{\theta^*\} \geq 0.5$. Then we get $K(\theta^*) = 0$ and $\theta^*(t) = 0 (\forall t \geq 0)$ and so

$$\mathrm{Cr}\{\theta^*\} = 0.5. \tag{3.3}$$

Note that by definition of Cr in (3.1), we have

$$\mathrm{Cr}\{\theta\} \leq 0.5 \ (\forall \theta \in H \text{ with } \theta \neq \theta^*). \tag{3.4}$$

We want to show that

$$\mathrm{Cr}\{\theta^*\} + \sup_{\substack{\theta\in H\\ \theta\neq\theta^*}} \mathrm{Cr}\{\theta\} = 1, \text{ i.e., } \sup_{\substack{\theta\in H\\ \theta\neq\theta^*}} \mathrm{Cr}\{\theta\} = 0.5 \tag{3.5}$$

holds. For any $\varepsilon (0 < \varepsilon < 1)$, define $\theta_\varepsilon \in H$ by

$$\theta_\varepsilon(t) = \begin{cases} \frac{\sqrt{6}}{\pi}(\log_e(\frac{1+\varepsilon}{1-\varepsilon}))t & \text{for } 0 \le t \le 1, \\ \frac{\sqrt{6}}{\pi}\log_e(\frac{1+\varepsilon}{1-\varepsilon}) & \text{for } 1 < t. \end{cases}$$

Then, we get

$$0.5 - \varepsilon < 0.5(1-\varepsilon) = \mathrm{Cr}\{\theta_\varepsilon\}. \tag{3.6}$$

Statements (3.4) and (3.6) say that $\mathrm{Cr}\{\theta^*\} + \sup_{\substack{\theta \in H \\ \theta \neq \theta^*}} \mathrm{Cr}\{\theta\} = 1$ holds for any $\theta^* \in H$ with $\mathrm{Cr}\{\theta^*\} \ge 0.5$. Therefore,

Lemma 3.1 $\mathrm{Cr}\{\theta\}(\theta \in H)$ *defined by* (3.1) *satisfies the (CEC).*

Let $\mathcal{P}$ be the power set of H. Through this function Cr on H and Theorem 2.2, we get a unique credibility measure Cr on $\mathcal{P}$ as in (2.2) of Theorem 2.2. Let $C(t,\theta) = C_t(\theta) = C_t$ be the fuzzy process defined by $C_t(\theta) = \theta(t)$ for any $\theta \in H$. We use the expression $\{C_{t_2} - C_{t_1} = x\}$ to denote the set $\{\theta \in H | \theta(t_2) - \theta(t_1) = x\}$.

Lemma 3.2

$$\mathrm{Cr}\{C_{t_2} - C_{t_1} = x\} = \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} \cdot \frac{|x|}{(t_2 - t_1)}\Big)\Big)^{-1}$$

holds for any x and $t_2 - t_1 > 0$.

Proof; Case $x \neq 0$. Set $A = \{\theta \in H | \theta(t_2) - \theta(t_1) = x\}$. For any $\theta \in A$, we get $K(\theta) \ge \frac{|x|}{t_2 - t_1}$ and

$$\mathrm{Cr}\{\theta\} \le \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} \cdot \frac{|x|}{(t_2 - t_1)}\Big)\Big)^{-1} < 0.5.$$

Hence we get

$$\sup_{\theta \in A} \mathrm{Cr}\{\theta\} \le \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} \cdot \frac{|x|}{(t_2 - t_1)}\Big)\Big)^{-1} < 0.5, \tag{3.7}$$

$$\mathrm{Cr}\{A\} = \sup_{\theta \in A} \mathrm{Cr}\{\theta\}. \tag{3.8}$$

Let $\theta_0(t) = \frac{x}{t_2 - t_1} t$. Then we get

$$\theta_0 \in A, \tag{3.9}$$

$$\mathrm{Cr}\{\theta_0\} = \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} \cdot \frac{|x|}{(t_2 - t_1)}\Big)\Big)^{-1}. \tag{3.10}$$

(3.7)&(3.8)&(3.9)&(3.10) imply that

$$\mathrm{Cr}\{A\} = \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} \cdot \frac{|x|}{(t_2 - t_1)}\Big)\Big)^{-1} \tag{3.11}$$

holds.
Case $x = 0$. Set $B = \{\theta \in H | \theta(t_2) - \theta(t_1) = 0\}$. Then we get

$$\sup_{\theta \in B} \mathrm{Cr}\{\theta\} = 0.5 \tag{3.12}$$

and so, through (2.2) in Theorem 2.2 we further have

$$\mathrm{Cr}\{B\} = 1 - \sup_{\theta \in B^c} \mathrm{Cr}\{\theta\}. \tag{3.13}$$

But $B^c = \cup_{x \neq 0} A_x$, where $A_x = \{\theta \in H | \theta(t_2) - \theta(t_1) = x\}, x \neq 0$, and so we get

$$\begin{aligned} \mathrm{Cr}\{B\} &= 1 - \sup_{x \neq 0} \sup_{\theta \in A_x} \mathrm{Cr}\{\theta\} \\ &= 1 - \sup_{x \neq 0} \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} \cdot \frac{|x|}{(t_2 - t_1)}\Big)\Big)^{-1} \text{ [by (3.11)]} \\ &= 1 - 0.5 = 0.5. \end{aligned} \tag{3.14}$$

This last equality (3.14) with (3.11) say that $C_{t_2} - C_{t_1}$ is stationary and is further a normally distributed fuzzy variable with expected value 0 and variance $t^2 = (t_2 - t_1)^2$ whose membership function is

$$2\Big(1 + \exp\Big(\frac{\pi |x|}{\sqrt{6} t}\Big)\Big)^{-1}, \ -\infty < x < \infty.$$

□

Let $t_0 = 0$ and $t_0 < t_1 < \cdots < t_n$ be given. Then we have

Theorem 3.1 *$C_{t_1} - C_{t_0}, \cdots, C_{t_n} - C_{t_{n-1}}$ are independent.*

Proof; We only have to prove

$$\mathrm{Cr}\{C_{t_1} - C_{t_0} = x_1 \,\&\cdots\&\, C_{t_n} - C_{t_{n-1}} = x_n\} = \bigwedge_{i=1}^{n} \mathrm{Cr}\{C_{t_i} - C_{t_{i-1}} = x_i\}$$

holds for any $x_1, \cdots, x_n$. Here $t_0 = 0$ and $C_{t_0}(\theta) = 0$ for any $\theta \in H$ and $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$ for any real numbers a, b. Let $A = \{\theta \in H | C_{t_i} - C_{t_{i-1}} = x_i (1 \leq i \leq n)\}$. For any $\theta \in A$, by definition of $K(\theta)$, we get

$$K(\theta) \geq \frac{|x_i|}{t_i - t_{i-1}}, \ K(\theta) \geq \bigvee_{i=1}^{n} \frac{|x_i|}{t_i - t_{i-1}}, \tag{3.15}$$

$$\mathrm{Cr}\{\theta\} = \Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}K(\theta)\Big)\Big)^{-1} \le \Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}\Big\{\bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}}\Big\}\Big)\Big)^{-1}. \tag{3.16}$$

Define $\theta_0(r)$ by

$$\theta_0(r) = \begin{cases} \dfrac{x_1}{t_1}r, & \text{if } 0 \le r < t_1 \\ \dfrac{x_k(r-t_{k-1})}{t_k-t_{k-1}} + \displaystyle\sum_{j=1}^{k-1} x_j, & \text{if } t_{k-1} \le r < t_k (2 \le k \le n) \\ \displaystyle\sum_{j=1}^{n} x_j, & \text{if } t_n \le r. \end{cases} \tag{3.17}$$

Then, we see that

$$\theta_0 \in A,\ K(\theta_0) \ge \bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}} \tag{3.18}$$

holds. In fact,

Lemma 3.3 $K(\theta_0) = \bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}}$.

Proof; Set $\bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}} = M(\ge 0)$. Let $0 \le s < t$ be such that $t_{j-1} \le s < t_j$ and $t_{k-1} \le t < t_k$, $1 \le j \le k \le n$. Using $-|x| \le x \le |x|$ for any number x and $\frac{|x_i|}{t_i-t_{i-1}} \le M(1 \le i \le n)$, we have

$$\begin{gathered} -(t_j-s)M \le \theta_0(t_j)-\theta_0(s) \le (t_j-s)M, \\ -(t_{j+1}-t_j)M \le \theta_0(t_{j+1})-\theta_0(t_j) \le (t_{j+1}-t_j)M, \\ \cdots \\ -(t_{k-1}-t_{k-2})M \le \theta_0(t_{k-1})-\theta_0(t_{k-2}) \le (t_{k-1}-t_{k-2})M, \\ -(t-t_{k-1})M \le \theta_0(t)-\theta_0(t_{k-1}) \le (t-t_{k-1})M. \end{gathered}$$

Summing up each sides in the above inequalities, we get

$$-M \le \frac{\theta_0(t)-\theta_0(s)}{t-s} \le M,\ t-s>0. \tag{3.19}$$

Definition of $K(\theta_0)$, and (3.19) lead to $K(\theta_0) \le M$. This last inequality with (3.18) shows that $K(\theta_0) = \bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}}$ holds. □

From here, argument breaks down into two cases.

Case I:There exists non zero $x_i(1 \le i \le n)$.

In this case, by (3.16) we get

$$\sup_{\theta\in A}\mathrm{Cr}\{\theta\} \le \Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}\Big\{\bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}}\Big\}\Big)\Big)^{-1} < 0.5 \tag{3.20}$$

and so by Theorem 2.2 we further have $\mathrm{Cr}\{A\} = \sup_{\theta\in A}\mathrm{Cr}\{\theta\}$. On the other hand, by Lemma 3.3 we have $K(\theta_0) = \bigvee_{i=1}^{n} \frac{|x_i|}{t_i-t_{i-1}}, \theta_0 \in A$ which say that

$$\sup_{\theta\in A}\mathrm{Cr}\{\theta\} = \Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}\Big\{\bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}}\Big\}\Big)\Big)^{-1}.$$

Therefore we get

$$\begin{aligned}\mathrm{Cr}\{C_{t_i}-C_{t_{i-1}} = x_i(1\le i\le n)\} = \mathrm{Cr}\{A\} &= \Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}\Big\{\bigvee_{i=1}^{n}\frac{|x_i|}{t_i-t_{i-1}}\Big\}\Big)\Big)^{-1}\\ &= \bigwedge_{i=1}^{n}\Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}\Big(\frac{|x_i|}{t_i-t_{i-1}}\Big)\Big)\Big)^{-1}\\ &= \bigwedge_{i=1}^{n}\mathrm{Cr}\{C_{t_i}-C_{t_{i-1}} = x_i\}[\text{by Lemma 3.2}]\end{aligned} \tag{3.21}$$

Case II: $x_1 = \cdots = x_n = 0$.
Let

$$B = \{C_{t_i}-C_{t_{i-1}} = 0\ (1\le i\le n)\} = \{\theta\in H|\theta(t_i)-\theta(t_{i-1}) = 0\ (1\le i\le n)\}.$$

Then, for any θ in B, we get $K(\theta)\ge 0$ with

$$\mathrm{Cr}\{\theta\} = \Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}K(\theta)\Big)\Big)^{-1} \le (1+\exp(0))^{-1} = 0.5. \tag{3.22}$$

Now, θ_0 at (3.17) becomes $\theta_0(t) = 0(\forall t\ge 0)$ and so $\theta_0\in B, K(\theta_0) = 0$. Therefore, we get $\sup_{\theta\in B}\mathrm{Cr}\{\theta\} = 0.5$. Hence $\mathrm{Cr}\{B\} = 1-\sup_{\theta\in B^c}\mathrm{Cr}\{\theta\}$, where

$$\begin{aligned}B^c &= \{C_{t_1}-C_{t_0}\ne 0 \text{ or } C_{t_2}-C_{t_1}\ne 0 \text{ or } \cdots \text{or } C_{t_n}-C_{t_{n-1}}\ne 0\}\\ &= \bigcup_{i=1}^{n}\Big\{\bigcup_{y_i\ne 0}\{C_{t_i}-C_{t_{i-1}} = y_i\}\Big\}\end{aligned}$$

and

$$\sup_{\theta\in B^c}\mathrm{Cr}\{\theta\} = \bigvee_{i=1}^{n}\sup_{y_i\ne 0,\theta}\big\{\mathrm{Cr}\{\theta\}|\theta\in H, \theta(t_i)-\theta(t_{i-1}) = y_i\big\}.$$

Set $D_{y_i} = \{\theta\in H|\theta(t_i)-\theta(t_{i-1}) = y_i\}$. Then for any $\theta\in D_{y_i}$, we get

$$\mathrm{Cr}\{\theta\} \le \Big(1+\exp\Big(\frac{\pi}{\sqrt{6}}\frac{|y_i|}{(t_i-t_{i-1})}\Big)\Big)^{-1} \text{ because } K(\theta)\ge \frac{|y_i|}{t_i-t_{i-1}}.$$

Setting $\theta_0(r) = \frac{y_i r}{t_i - t_{i-1}}$, we get $\theta_0 \in D_{y_i}$ and $K(\theta_0) = \frac{|y_i|}{t_i - t_{i-1}}$ and so we get

$$\begin{aligned}\sup_{y_i \neq 0, \theta} \left\{ \mathrm{Cr}\{\theta\} | \theta \in H, \theta(t_i) - \theta(t_{i-1}) = y_i \right\} &= \sup_{y_i \neq 0} \sup_{\theta \in D_{y_i}} \mathrm{Cr}\{\theta\} \\ &= \sup_{y_i \neq 0} \Big(1 + \exp\Big(\frac{\pi}{\sqrt{6}} \frac{|y_i|}{(t_i - t_{i-1})} \Big) \Big)^{-1} \\ &= (1 + \exp(0))^{-1} = 0.5.\end{aligned}$$

Therefore, we finally get $\sup_{\theta \in B^c} \mathrm{Cr}\{\theta\} = \bigvee_{i=1}^n 0.5 = 0.5, \mathrm{Cr}\{B\} = 1 - \sup_{\theta \in B^c} \mathrm{Cr}\{\theta\} = 0.5$,

$$\mathrm{Cr}\{C_{t_i} - C_{t_{i-1}} = 0 \ (1 \leq i \leq n)\} = \bigwedge_{i=1}^n \mathrm{Cr}\{C_{t_i} - C_{t_{i-1}} = 0\}. \tag{3.23}$$

Combining (3.21) & (3.23) we have got that $C_{t_1} - C_{t_0}, C_{t_2} - C_{t_1}, \cdots, C_{t_n} - C_{t_{n-1}}$ are independent. □

Lemma 3.2 and Theorem 3.1 show us the existence of standard Liu process.

4 Construction of Liu Process

Let C_t be a standard Liu process. Then, by definition we have

$$\mathrm{Cr}\{C_t = x\} = \Big(1 + \exp\Big(\frac{\pi |x|}{\sqrt{6} t} \Big) \Big)^{-1}, \ -\infty < x < \infty. \tag{4.1}$$

For an arbitrary positive number σ and any number e, define $\xi_t = \sigma C_t + et$. Then we get $\xi_0 = \sigma C_0 + e \cdot 0 = 0$ because$C_0 = 0$. Here, we further have

$$\mathrm{Cr}\{\xi_t = u\} = \mathrm{Cr}\{C_t = \frac{u - et}{\sigma}\} = \Big(1 + \exp\Big(\frac{\pi |u - et|}{\sqrt{6} \sigma t} \Big) \Big)^{-1}, \ -\infty < u < \infty. \tag{4.2}$$

Let $t_0 = 0 < t_1 < t_2 < \cdots < t_n$. Then, we get

$$\begin{aligned}\mathrm{Cr}\{\xi_{t_i} - \xi_{t_{i-1}} = u_i (1 \leq i \leq n)\} &= \mathrm{Cr}\{C_{t_i} - C_{t_{i-1}} = \frac{u_i - e(t_i - t_{i-1})}{\sigma} (1 \leq i \leq n)\} \\ &= \bigwedge_{i=1}^n \mathrm{Cr}\{C_{t_i} - C_{t_{i-1}} = \frac{u_i - e(t_i - t_{i-1})}{\sigma}\} \\ &\quad [\text{by independence of } C_{t_i} - C_{t_{i-1}}, 1 \leq i \leq n] \\ &= \bigwedge_{i=1}^n \mathrm{Cr}\{\xi_{t_i} - \xi_{t_{i-1}} = u_i\}.\end{aligned} \tag{4.3}$$

Hence,

Theorem 4.1 *$\xi_{t_1} - \xi_{t_0}, \cdots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent.*

Furthermore, we get

$$\begin{aligned}\mathrm{Cr}\{\xi_{t_i} - \xi_{t_{i-1}} = u_i\} &= \mathrm{Cr}\{C_{t_i} - C_{t_{i-1}} = \frac{u_i - e(t_i - t_{i-1})}{\sigma}\} \\ &= \Big(1 + \exp\Big(\frac{\pi|u_i - e(t_i - t_{i-1})|}{\sqrt{6}\sigma(t_i - t_{i-1})}\Big)\Big)^{-1}.\end{aligned} \tag{4.4}$$

Hence, ξ_t is stationary. Note that $\mathrm{E}\xi_t = et$ and $\mathrm{Var}\xi_t = \sigma^2 t^2$.Therefore ξ_t is a Liu process.

5 Conclusion

We have constructed Liu process in an exact way.

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