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On Classification of Graphs by Chromatic Polynomials

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Abstract

For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. In this paper, we discuss a pair of chromatically equivalent of K_4-homeomorphs with girth 8, that is, $K_4(1, 3, 4, d, e, f)$ and $K_4(1, 3, 4, d', e', f')$. As a result, we obtain two infinite chromatically equivalent non-isomorphic K_4-homeomorphs.

Mathematical Subject Classification: Primary 05C15

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1 Introduction

In graph theory, graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called color to elements of a graph subject to certain constraints. It is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color and this is called a vertex coloring. The chromatic polynomial of a graph counts the number of its proper vertex colorings. One of the applications of vertex coloring of a graph is scheduling problems. Vertex coloring models to a number of scheduling problems. In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduled in any order, but pairs of jobs may be in conflict in the sense that they may not be assigned to the same slot. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The chromatic number of the graph is exactly the minimum makespan, the optimal time to finish all jobs without conflicts.

All graphs considered here are simple graphs. For such a graph G, let $P(G,\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G.

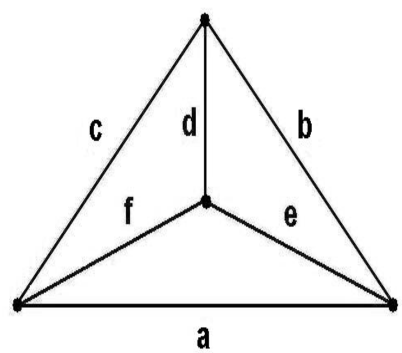

Figure 1: $K_4(a,b,c,d,e,f)$

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of K_4 are replaced by the six paths of length a,b,c,d,e,f, respectively, as shown in Figure 1. So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \leq g \leq 7$ has been studied by many authors (see [4,6,7,8]). The chromaticity of K_4-homeomorphs with girth 8 is still remains open. When referring to the chromaticity of K_4-homeomorphs with girth 8, we know that only five types of K_4-homeomorphs with girth 8 need to be solved, i.e., $K_4(1,2,5,d,e,f)$, $K_4(1,3,4,d,e,f)$, $K_4(2,3,3,d,e,f)$, $K_4(2,4,2,d,e,f)$ and $K_4(1,2,2,3,e,f)$. For some results on chromatic equivalence of K_4-homeomorphs with girth 10, please refer [1,2].

In [9,10], YanIing Peng characterized two chromatically equivalence pairs of $K_4(1,2,5,d,e,f)$ and $K_4(1,2,5,d',e',f')$; and $K_4(2,3,3,d,e,f)$ and $K_4(2,3,3,d',e',f')$. In this paper, we shall discuss a chromatically equivalence pair of K_4-homeomorphs, $K_4(1,3,4,d,e,f)$ (as in Figure 2) and $K_4(1,3,4,d',e',f')$. We obtain two infinite chromatically equivalent non-isomorphic K_4-homeomorphs. This result

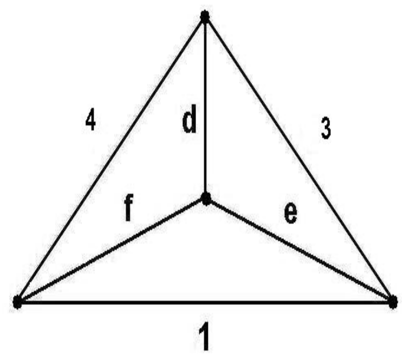

Figure 2: $K_4(1,3,4,d,e,f)$

can be extended to the study of chromatic equivalence classes of $K_4(1,3,4,d,e,f)$ and chromatic uniqueness of K_4-homeomorphs with girth 8.

2 Preliminary Results

In this section, we give some known results used in the sequel.

Lemma 2.1 *Assume that G and H are $\chi-$equivalent. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see [5]);*

(2) *G and H has the same girth and same number of cycles with length equal to their girth (see [12]);*

(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph (see [3]);*

(4) *Let $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$, then*

(i) *$min\,(a,b,c,d,e,f) = min\,(a',b',c',d',e',f')$ and the number of times that this minimum occurs in the list $\{a,b,c,d,e,f\}$ is equal to the number of times that this minimum occurs in the list $\{a',b',c',d',e',f'\}$ (see[11]);*

(ii) *if* $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ *as multisets, then* $H \cong G$ *(see [6]).*

Theorem 2.1 *(Sabina et al. [1]) Let* $K_4(1, 2, 7, d, e, f)$ *and* $K_4(1, 2, 7, d', e', f')$ *be chromatically equivalent, then*

$$K_4(1, 2, 7, i, i+8, i+1) \sim K_4(1, 2, 7, i, i+2, i+7),$$
$$K_4(1, 2, 7, i, i+1, i+8) \sim K_4(1, 2, 7, i+7, i, i+2),$$
$$K_4(1, 2, 7, i, i+1, i+3) \sim K_4(1, 2, 7, i+2, i+2, i).$$

Theorem 2.2 *(Sabina et al. [2]) Let* $K_4(1, 3, 6, d, e, f)$ *and* $K_4(1, 3, 6, d', e', f')$ *be chromatically equivalent, then*

$$K_4(1, 3, 6, i, i+1, i+4) \sim K_4(1, 3, 6, i+2, i+3, i),$$
$$K_4(1, 3, 6, i, i+7, i+1) \sim K_4(1, 3, 6, i+2, i, i+6).$$

Theorem 2.3 *(YanIing Peng [9]) Let* $K_4(1, 2, 5, d, e, f)$ *and* $K_4(1, 2, 5, d', e', f')$ *be chromatically equivalent, then*

$$K_4(1, 2, 5, i, i+6, i+1) \sim K_4(1, 2, 5, i+2, i, i+5),$$
$$K_4(1, 2, 5, i, i+1, i+6) \sim K_4(1, 2, 5, i+5, i, i+2),$$
$$K_4(1, 2, 5, i, i+1, i+3) \sim K_4(1, 2, 5, i+2, i+2, i).$$

Theorem 2.4 *(YanIing Peng [10]) Let* $K_4(2, 3, 3, d, e, f)$ *and* $K_4(2, 3, 3, d', e', f')$ *be chromatically equivalent, then* $K_4(2, 3, 3, d, e, f)$ *is isomorphic to* $K_4(2, 3, 3, d', e', f')$.

3 Main Results

In this section, we present our main results.

Lemma 3.1 *Assume that* $G \cong K_4(1, 3, 4, d, e, f)$ *and* $H \cong K_4(1, 3, 4, d', e', f')$, *then*

(1) $P(G) = (-1)^{x-1}[s/(s-1)^2][-s^{x-1} - s^8 - s^7 - s^3 - s^2 + 2s + 2 + R(G)]$, *where*

$R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+7} + s^{f+8} + s^{d+9} + s^{d+e+f}$,

$s = 1 - \lambda$, *x is the number of the edges of G.*

(2) *If* $P(G) = P(H)$, *then* $R(G) = R(H)$.

Proof. (1) Let $s = 1 - \lambda$. From [11], we have the chromatic polynomial of K_4-homeomorphs $K_4(a, b, c, d, e, f)$ is as follows:

$$P(K_4(a,b,c,d,e,f)) = (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2)-(s+1)(s^a+s^b+s^c+s^d+s^e+s^f)+(s^{a+d}+s^{b+f}+s^{c+e}+s^{a+b+e}+s^{b+d+c}+s^{a+c+f}+s^{d+e+f}-s^{x-1})].$$

So when $a = 1, b = 3$ and $c = 4$, we have

$$\begin{aligned} P(K_4(1,3,4,d,e,f)) &= (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2)-(s+1)(s+s^3+s^4+s^d+s^e+s^f)+(s^{1+d}+s^{3+f}+s^{4+e}+s^{4+e}+s^{7+d}+s^{5+f}+s^{d+e+f}-s^{x-1})] \\ &= (-1)^{x-1}[s/(s-1)^2][-s^{x-1}-s^5-2s^4-s^3+2s+2+R(G)] \end{aligned}$$

where

$R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{f+5} + s^{d+7} + s^{d+e+f}$

as required.

(2) If $P(G) = P(H)$, then we can easily see that $R(G) = R(H)$. .

Theorem 3.1 *Let K_4-homeomorphs $K_4(1,3,4,d,e,f)$ and $K_4(1,3,4,d',e',f')$ be chromatically equivalent, then we have*

$$K_4(1,3,4,i,i+5,i+1) \sim K_4(1,3,4,i+2,i,i+4),$$
$$K_4(1,3,4,i,i+1,i+4) \sim K_4(1,3,4,i+2,i+3,i).$$

where $i \geq 1$.

Proof. Assume that $G \cong K_4(1,3,4,d,e,f)$ and $H \cong K_4(1,3,4,d',e',f')$. We now solve for the equation $R(G) = R(H)$ to find G and H which are not isomorphic. From Lemma 3.1, we have

$R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{f+5} + s^{d+7} + s^{d+e+f}$,
$R(H) = -s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{f'+3} + 2s^{e'+4} + s^{f'+5} + s^{d'+7} + s^{d'+e'+f'}$.

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1 (1), $d+e+f=d'+e'+f'$. We obtain the following after simplification. Note that our assumption in the following steps of the proof is $R_j(G)=R_j(H)$, where $1\leq j\leq 19$.

$$R_1(G)=-s^d-s^e-s^f-s^{e+1}-s^{f+1}+s^{f+3}+2s^{e+4}+s^{f+5}+s^{d+7},$$
$$R_1(H)=-s^{d'}-s^{e'}-s^{f'}-s^{e'+1}-s^{f'+1}+s^{f'+3}+2s^{e'+4}+s^{f'+5}+s^{d'+7}.$$

Let us consider the h.r.p. in $R_1(G)$ and the h.r.p. in $R_1(H)$. We have max $\{e+4, f+5, d+7\}=$ max $\{e'+4, f'+5, d'+7\}$. Without loss of generality, we will consider only the following six cases.

Case 1. If max $\{e+4, f+5, d+7\}=e+4$ and max $\{e'+4, f'+5, d'+7\}$ $=e'+4$, then $e=e'$. Thus, we can cancel the following pairs of terms in the equations : $-s^e$ with $-s^{e'}$, $-s^{e+1}$ with $-s^{e'+1}$ and $2s^{e+4}$ with $2s^{e'+4}$. Therefore, the l.r.p. in $R_1(G)$ is d or f and the l.r.p. in $R_1(H)$ is d' or f'. So, $d=f'$ or $d=d'$ or $f=f'$ or $f=d'$. We have $e=e'$ and $d+e+f=d'+e'+f'$. So, we know that $\{d,e,f\}=\{d',e',f'\}$ as multisets. From Lemma 2.1 (4(ii)), $G\cong H$.

Case 2. If max $\{e+4, f+5, d+7\}=f+5$ and max $\{e'+4, f'+5, d'+7\}=$ $f'+5$, then $f=f'$. We can deal with this case in the same way as Case 1, thus, $G\cong H$.

Case 3. If max $\{e+4, f+5, d+7\}=d+7$ and max $\{e'+4, f'+5, d'+7\}=$ $d'+7$, then we can deal with this case in the same way as Case 1. So, we have $G\cong H$.

Case 4. If max $\{e+4, f+5, d+7\}=e+4$ and max $\{e'+4, f'+5, d'+7\}=$ $f'+5$, then $e+4=f'+5$, that is

$$f'=e-1 \quad (1)$$

from $d+e+f=d'+e'+f'$, we have

$$d+f=d'+e'-1. \quad (2)$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. From Lemma 2.1(4(i)), min $\{d,e,f\}=$ min $\{d',e',f'\}$. Without loss of generality, let min $\{d,e,f\}=d$. The following subcases need to be considered.

Subcase 4.1. If min $\{d,e,f\}=d$ and min $\{d',e',f'\}=d'$, then $d=d'$. Thus, we can consider this case the same way as Case 1. So, $G\cong H$.

Subcase 4.2. If min $\{d,e,f\}=d$ and min $\{d',e',f'\}=e'$, then $d=e'$. From Equation (2), we have $d'=f+1$. Note that $f'=e-1$ by Equation (1). We can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_2(G)=-s^d-s^e-s^f-s^{e+1}-s^{f+1}+s^{f+3}+2s^{e+4}+s^{f+5}+s^{d+7},$$
$$R_2(H)=-s^{f+1}-s^d-s^{e-1}-s^{d+1}-s^e+s^{e+2}+2s^{d+4}+s^{e+4}+s^{f+8}.$$

After simplifying $R_2(G)$ and $R_2(G)$, we have

$R_3(G) = -s^f - s^{e+1} + s^{f+3} + s^{e+4} + s^{f+5} + s^{d+7}$,
$R_3(H) = -s^{e-1} - s^{d+1} + s^{e+2} + 2s^{d+4} + s^{f+8}$.

Consider the term $-s^{d+1}$ in $R_3(H)$. Since the min $\{d, e, f\} = d$, $-s^{d+1}$ in $R_3(H)$ cannot be cancelled by any of the positive terms in $R_3(H)$. Thus, $-s^{d+1}$ must be equal to $-s^f$ or $-s^{e+1}$ in $R_3(G)$. Note that max $\{e+4, f+5, d+7\} = e+4$, so $e+4 \geq d+7$, that is, $e+1 \geq d+4 > d+1$. Thus, $-s^{e+1} \neq -s^{d+1}$. If $-s^{d+1} = -s^f$, then $d+1 = f$. Thus, $R_3(G)$ and $R_3(H)$ can be written as follows:

$R_4(G) = -s^{d+1} - s^{e+1} + s^{d+4} + s^{e+4} + s^{d+6} + s^{d+7}$,
$R_4(H) = -s^{e-1} - s^{d+1} + s^{e+2} + 2s^{d+4} + s^{d+9}$.

After simplifying $R_4(G)$ and $R_4(H)$, we have

$R_5(G) = -s^{e+1} + s^{e+4} + s^{d+6} + s^{d+7}$,
$R_5(H) = -s^{e-1} + s^{e+2} + s^{d+4} + s^{d+9}$.

Thus, we have

$$-s^{e+1} + s^{e+4} + s^{d+6} + s^{d+7} = -s^{e-1} + s^{e+2} + s^{d+4} + s^{d+9}.$$

Therefore, we have $e = d+5$. At this point, we acquire the following equations : $e = d+5$, $f' = e-1 = d+4$, $d' = f+1 = d+2$ and $e' = d$. Let $d = i$. Therefore, we obtain the solution where G is isomorphic to $K_4(1, 3, 4, i, i+5, i+1)$ and H is isomorphic to $K_4(1, 3, 4, i+2, i, i+4)$.

Subcase 4.3. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. Note that max $\{e'+4, f'+5, d'+7\} = f'+5$. So, $f'+5 \geq d'+7$, that is, $f' \geq d'+2 > d'$. This contradicts with the fact that min $\{d', e', f'\} = f'$.

Case 5. If max $\{e+4, f+5, d+7\} = e+4$ and max $\{e'+4, f'+5, d'+7\} = d'+7$, then $e+4 = d'+7$, that is,

$$d' = e - 3 \tag{3}$$

from $d+e+f = d'+e'+f'$, we have

$$d + f + 3 = e' + f'. \tag{4}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$, where min $\{d, e, f\}$ = min $\{d', e', f'\}$ by Lemma 2.1(4(i)). Without loss of generality, let min $\{d, e, f\} = d$. The following subcases need to be considered.

Subcase 5.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then we deal with this case the same way with Case 1. So, we get $G \cong H$.

Subcase 5.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From Equation (4), we have $f' = f+3$. Note that $d' = e-3$ by Equation (3). Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$R_6(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{f+5} + s^{d+7}$,

$$R_6(H) = -s^{e-3} - s^{d} - s^{f+3} - s^{d+1} - s^{f+4} + s^{f+6} + 2s^{d+4} + s^{f+8} + s^{e+4}.$$

Consider the term $-s^{d+1}$ in $R_6(H)$. Since min $(d, e, f) = d$, then $-s^{d+1}$ cannot be cancelled by any positive terms in $R_6(H)$. Note that max $\{e+4, f+5, d+7\} = e+4$, so $e+4 \geq d+7$, that is $e+1 \geq d+4 > d+1$, thus $e+1 \neq d+1$. Moreover $e \geq d+3 > d+1$, thus $e \neq d+1$. Therefore the term $-s^{d+1}$ in $R_6(H)$ must be cancelled by the term $-s^{f}$ or $-s^{f+1}$ in $R_6(G)$.

If $-s^{f} = -s^{d+1}$, then $f = d+1$. Thus, $R_6(G)$ and $R_6(H)$ can be written as follows:

$$R_7(G) = -s^{e} - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+6} + s^{d+7},$$
$$R_7(H) = -s^{e-3} - s^{d+4} - s^{d+1} - s^{d+5} + s^{d+7} + 2s^{d+4} + s^{d+9}.$$

After simplifying, we obtain

$$-s^{e} - s^{e+1} - s^{d+2} - s^{e+4} + s^{d+6} = -s^{e-3} - s^{d+5} + s^{d+9}.$$

Therefore, we have $e-3 = d+2$. At this point, we acquire the following equations: $e = d+5$, $f = d+1$, $e' = d$, $f' = f+3 = d+4$ and $d' = e-3 = d+2$. Let $d = i$. Hence, we obtain the solution where G is isomorphic to $K_4(1, 3, 4, i, i+5, i+1)$ and H is isomorphic to $K_4(1, 3, 4, i+2, i, i+4)$. Note that this is the same solution as in Subcase 4.2.

If $-s^{f+1} = -s^{d+1}$, then $f = d$. Thus, $R_6(G)$ and $R_6(H)$ can be written as follows:

$$R_8(G) = -s^{e} - s^{d} - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+5} + s^{d+7},$$
$$R_8(H) = -s^{e-3} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+6} + 2s^{d+4} + s^{d+8}.$$

After simplifying, we obtain

$$-s^{e} - s^{d} - s^{e+1} - s^{d+3} + s^{e+5} + s^{d+5} + s^{d+7} = -s^{e-3} - s^{d+3} + s^{d+6} + s^{d+4} + s^{d+8}.$$

The resulting equation contradicts $R_8(G) = R_8(H)$.

Subcase 5.3. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. From Equation (4), $e' = f+3$. Note that from Equation (3), we have $d' = e-3$. We can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_9(G) = -s^{d} - s^{e} - s^{f} - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{f+5} + s^{d+7},$$
$$R_9(H) = -s^{e-3} - s^{f+3} - s^{d} - s^{f+4} - s^{d+1} + s^{d+3} + 2s^{f+7} + s^{d+5} + s^{e+4}.$$

For the same reasons stated by Subcase 5.2, $-s^{d+1}$ in $R_9(H)$ must be equal to $-s^{f}$ or $-s^{f+1}$ in $R_9(G)$. If $-s^{d+1} = -s^{f}$, then $d+1 = f$. We can write $R_9(G)$ and $R_9(H)$ as follows:

$$R_{10}(G) = -s^{e} - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+6} + s^{d+7},$$
$$R_{10}(H) = -s^{e-3} - s^{d+4} - s^{d+5} - s^{d+1} + s^{d+3} + 2s^{d+8} + s^{d+5}.$$

After simplifying, we have

$$-s^e - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+6} + s^{d+7} = -s^{e-3} - s^{d+4} + s^{d+3} + 2s^{d+8}.$$

The resulting equation contradicts $R_{10}(G) = R_{10}(H)$.

If $-s^{d+1} = -s^{f+1}$, we have $d = f$. Thus, we have the following equations as follows:

$$R_{11}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+5} + s^{d+7},$$
$$R_{11}(H) = -s^{e-3} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + 2s^{d+7} + s^{d+5}.$$

After simplifying, we have

$$-s^e - s^d - s^{e+1} - s^{d+1} + s^{e+4} = -s^{e-3} - s^{d+3} - s^{d+4} + s^{d+7}.$$

So, we obtain $e - 3 = d$ by min $\{d, e, f\} = d$. We then have $e = d + 3$, $f = d = f'$, $e' = f + 3 = d + 3 = e$ and $d' = e - 3 = d$. Thus, $e = e'$, $d = d'$ and $f = f'$. Hence, $G \cong H$.

Case 6. If max $\{e+4, f+5, d+7\} = f+5$ and max $\{e'+4, f'+5, d'+7\} = d'+7$, then $f + 5 = d' + 7$, that is,

$$d' = f - 2 \tag{5}$$

from $d + e + f = d' + e' + f'$, we have

$$d + e = e' + f' - 2. \tag{6}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. We have min $\{d, e, f\}$ = min $\{d', e', f'\}$ by Lemma 2.1(4(i)). Without loss of generality, let min $\{d, e, f\} = d$. The following subcases need to be considered.

Subcase 6.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then we deal with this case the same way with Case 1. So, we get $G \cong H$.

Subcase 6.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From Equation (6), we have $f' = e + 2$. Note that $d' = f - 2$ by Equation (5). Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_{12}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{f+5} + s^{d+7},$$
$$R_{12}(H) = -s^{f-2} - s^d - s^{e+2} - s^{d+1} - s^{e+3} + s^{e+5} + 2s^{d+4} + s^{e+7} + s^{f+5}.$$

After simplifying, we have

$$R_{13}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{d+7},$$
$$R_{13}(H) = -s^{f-2} - s^{e+2} - s^{d+1} - s^{e+3} + s^{e+5} + 2s^{d+4} + s^{e+7}.$$

Consider the term $-s^{d+1}$ in $R_{13}(H)$. Since min $\{d, e, f\} = d$, $-s^{d+1}$ cannot be cancelled by any positive term in $R_{13}(H)$. From max $\{e+4, f+5, d+7\} = f+5$, we have $f + 5 \geq d + 7$, i.e., $f + 1 \geq d + 3 > d + 1$. So, $f + 1 \neq d + 1$. Moreover,

$f \geq d+2 > d+1$, thus, $f \neq d+1$, i.e., $-s^f \neq -s^{d+1}$. So, $-s^{d+1}$ in $R_{13}(H)$ must be equal to $-s^e$ or $-s^{e+1}$ in $R_{13}(G)$.

If $-s^{d+1} = -s^e$, then $e = d+1$. So, we have

$$R_{14}(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{f+3} + 2s^{d+5} + s^{d+7},$$
$$R_{14}(H) = -s^{f-2} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+6} + 2s^{d+4} + s^{d+8}.$$

After simplifying, we have

$$-s^f - s^{d+2} - s^{f+1} + s^{f+3} + 2s^{d+5} + s^{d+7} = -s^{f-2} - s^{d+3} + s^{d+6} + s^{d+4} + s^{d+8}.$$

The resulting equation contradicts $R_{14}(G) = R_{14}(H)$.

If $-s^{d+1} = -s^{e+1}$, then $d = e$. Thus, we have

$$R_{15}(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + 2s^{d+4} + s^{d+7},$$
$$R_{15}(H) = -s^{f-2} - s^{d+2} - s^{d+1} - s^{d+3} + s^{d+5} + 2s^{d+4} + s^{d+7}.$$

After simplifying, we have

$$-s^d - s^f - s^{f+1} + s^{f+3} = -s^{f-2} - s^{d+2} - s^{d+3} + s^{d+5}.$$

Since min $\{d, e, f\} = d$, then the term $-s^d$ must be equal to the term $-s^{f-2}$. So we have $d = f-2$. Thus we have the following equations: $f = d+2$, $d = e = e'$ and $f' = e+2 = d+2 = f$. From $d+e+f = d'+e'+f'$, we have $d = d'$ and hence $G \cong H$.

Subcase 6.3. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. From Equation (6), we have $e' = e+2$. Note that $d' = f-2$ by Equation (5). Thus, we have

$$R_{16}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{f+5} + s^{d+7},$$
$$R_{16}(H) = -s^{f-2} - s^{e+2} - s^d - s^{e+3} - s^{d+1} + s^{d+3} + 2s^{e+6} + s^{d+5} + s^{f+5}.$$

After simplifying, we have

$$R_{17}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + 2s^{e+4} + s^{d+7},$$
$$R_{17}(H) = -s^{f-2} - s^{e+2} - s^{e+3} - s^{d+1} + s^{d+3} + 2s^{e+6} + s^{d+5}.$$

For the same reasons stated in Subcase 6.2, $-s^{d+1}$ in $R_{17}(H)$ can only be equal to $-s^e$ or $-s^{f+1}$ in $R_{17}(G)$.

If $-s^{d+1} = -s^e$, then $d+1 = e$. So, we have

$$R_{18}(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{f+3} + 2s^{d+5} + s^{d+7},$$
$$R_{18}(H) = -s^{f-2} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + 2s^{d+7} + s^{d+5}.$$

After simplifying, we have

$$-s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} = -s^{f-2} - s^{d+4} + s^{d+7}.$$

Then, we know that the term s^{d+2} must be equal to s^{f-2}. So, we have $f = d+4$. Thus, we obtain the following equations: $f = d+4$, $d' = f-2 = d+2$, $f' = d$ and $e' = e+2 = d+3$. Let $d = i$. Hence, we obtain the solution where G is isomorphic to $K_4(1,3,4,i,i+1,i+4)$ and H is isomorphic to $K_4(1,3,4,i+2,i+3,i)$.

If $-s^{d+1} = -s^{e+1}$, then $d = e$. So, we have

$$R_{19}(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + 2s^{d+4} + s^{d+7},$$
$$R_{19}(H) = -s^{f-2} - s^{d+2} - s^{d+3} - s^{d+1} + s^{d+3} + 2s^{d+6} + s^{d+5}.$$

After simplifying, we obtain

$$-s^d - s^f - s^{f+1} + s^{f+3} + 2s^{d+4} + s^{d+7} = -s^{f-2} - s^{d+2} + 2s^{d+6} + s^{d+5}.$$

The resulting equation contradicts $R_{19}(G) = R_{19}(H)$.

At this point, we have solved the equation $R(G) = R(H)$ and the solution is as follows:

$$K_4(1,3,4,i,i+5,i+1) \sim K_4(1,3,4,i+2,i,i+4),$$
$$K_4(1,3,4,i,i+1,i+4) \sim K_4(1,3,4,i+2,i+3,i),$$

where $i \geq 1$. The proof is now completed.

We close the paper with the following problem.

Problem Study the chromatic uniqueness of the graph $K_4(1,3,4,d,e,f)$, where $d+e \geq 5$, $d+f \geq 4$ and $e+f \geq 7$.

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