Applied Mathematical Sciences, Vol. 6, 2012, no. 60, 2957 - 2962

Nonlinear Oscillation of a Class of Second-Order Dynamic Equations on Time Scales

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Abstract

In this paper, we discuss the oscillation of certain second-order nonlinear dynamic equations on time scales. We establish some oscillation criteria for the equations by applying a generalized Riccati transformation technique. Our results improve and extend some known results in the recent literature.

Mathematics Subject Classification: 39A10

Keywords: Nonlinear oscillation, Dynamic equation, Time scale, Riccati transformation

1 Introduction

The paper is to deal with the oscillation of the following second-order nonlinear dynamic equation

$$\left(r(t)\,|\,x^{\Delta}(t)\,|^{\alpha-1}\,x^{\Delta}(t)\right)^{\Delta}+q(t)\,|\,x(t)\,|^{\beta-1}\,x(t)=0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where the following conditions are assumed to hold: (H_1) α, β > 0 are constants and $\sup\mathbb{T}=\infty$; (H_2) r and q are positive rd-continuous functions defined on the time scale interval $[t_0,\infty)$.

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. Thesis [7] in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ (see [5]). Not only can the theory of dynamic equations on time scales unify the theories of differential equations and difference equations, but it is also able to extend these classical cases to cases ‘‘in between,” e.g., to the so-called q-difference equations. Dynamic equations on time scales have a lot of applications in population dynamics, quantum mechanics, electrical engineering, neural networks, heat transfer, combinatorics and so on. The book on the subject of time scales by Bohner and Peterson [5] summarize and organize much of time scale calculus and some applications.

By a solution of (1.1), we mean a nontrivial real function x such that $x \in C_{rd}^{1}[t_x, \infty)$ and $r(t)|x^{\Delta}(t)|^{\alpha-1} x^{\Delta}(t) \in C_{rd}^{1}[t_x, \infty)$ for a certain $t_x \geq t_0$ and satisfying (1.1) for $t \geq t_x$. Our attention is restricted to those solutions of (1.1) which exist on the half-line $[t_x, \infty)$ and satisfy $\sup\{|x(t)|: t > t_*\} > 0$ for any $t_* \geq t_x$. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In the last decade, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [1-4, 6, 8-10] and the references cited therein.

Recently, Saker [8] established some oscillation criteria for the second-order half-linear dynamic equation

$$\left(r(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta} + q(t)x^{\alpha}(t) = 0 \tag{1.2}$$

on time scales, where $\alpha > 1$ is an odd positive integer, and r and q satisfy (H_2). Afterward, Hassan [10] also considered (1.2), where α is a quotient of odd positive integers, and obtained some sufficient conditions for the oscillation of the equation. Hassan [10] improved and extended the results of Saker [8]. Very recently, Grace et al. [9] studied the oscillation of the second-order nonlinear dynamic equation

$$\left(r(t)(x^{\Delta}(t))^{\alpha}\right)^{\Delta} + q(t)x^{\beta}(t) = 0 \tag{1.3}$$

on time scales, where α, β are quotients of odd positive integers, and r and q satisfy (H_2). Grace et al. [9] obtained some new oscillation results for (1.3) when $\alpha < \beta$, $\alpha = \beta$ and $\alpha > \beta$, respectively.

It is clear that (1.2) and (1.3) are special cases of (1.1), and all the results of Saker [8], Hassan [10] and Grace et al. [9] can not be applied to (1.1) when α, β are not equal to quotients of odd positive integers. Therefore, it is of great interest to study the oscillation of (1.1) when α, $\beta > 0$ are constants. In this paper, we will establish some new oscillation criteria for (1.1) when α, $\beta > 0$ are constants. Our results extend and improve the results of Saker [8], Hassan [10] and Grace et al. [9].

We will need the following lemma to prove our main results.

Lemma 1.1. (Bohner and Peterson [5], p. 32, Theorem 1.87) Let $f:\mathbb{R}\to\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\to\mathbb{R}$ is delta differentiable and satisfies

$$(f\circ g)^{\Delta}(t)=\left\{\int_0^1 f'\left(g(t)+h\mu(t)g^{\Delta}(t)\right)dh\right\}g^{\Delta}(t),$$

where $\mu(t):=\sigma(t)-t$ is the graininess function on $\mathbb{T}$, here $\sigma(t):=\inf\{s\in\mathbb{T}\, s>t\}$ is the forward jump operator on $\mathbb{T}$.

2 Main Results

Theorem 2.1. *Suppose that (H$_1$), (H$_2$) and the following condition hold:*

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\Delta t=\infty. \tag{2.1}$$

Furthermore, assume that there exists a positive nondecreasing delta differentiable function φ such that for all $T>t_1\ge t_0$,

$$\limsup_{t\to\infty}\int_T^t\left[q(s)\varphi(s)-\varphi^{\Delta}(s)u^{\alpha}(s)v(s)\right]\Delta s=\infty, \tag{2.2}$$

where $u(t):=\left(\int_{t_1}^t r^{-1/\alpha}(s)\Delta s\right)^{-1}$ *and*

$$v(t):=\begin{cases} c_1, & c_1 \textit{ is any positive constant}, \textit{ if } \alpha<\beta,\\ 1, & \textit{if } \alpha=\beta,\\ c_2u^{\beta-\alpha}(t), & c_2 \textit{ is any positive constant, if } \alpha>\beta.\end{cases}$$

Then (1.1) is oscillatory.

Proof. Suppose that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1). Then there exists $t_1\ge t_0$ such that $x(t)>0$ for $t\in[t_1,\infty)$. Therefore, from (1.1) we have $(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta}=-q(t)x^{\beta}(t)<0$ for $t\in[t_1,\infty)$. It is easy to see that $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)$ is strictly decreasing on $[t_1,\infty)$ and is eventually of one sign. We claim $x^{\Delta}(t)>0$ for $t\in[t_1,\infty)$. Assume on the contrary, then there exists $t_2\ge t_1$ such that $x^{\Delta}(t_2)\le 0$. Take $t_3>t_2$, then we obtain $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)\le r(t_3)|x^{\Delta}(t_3)|^{\alpha-1}x^{\Delta}(t_3):=M<r(t_2)|x^{\Delta}(t_2)|^{\alpha-1}x^{\Delta}(t_2)\le 0$ for $t\in[t_3,\infty)$. Hence, we get $x^{\Delta}(t)\le-(-M)^{1/\alpha}r^{-1/\alpha}(t)$ for $t\in[t_3,\infty)$. Integrating both sides of the last inequality from t_3 to t, we have $x(t)-x(t_3)\le-(-M)^{1/\alpha}\int_{t_3}^t r^{-1/\alpha}(s)\Delta s$ for $t\in[t_3,\infty)$. Letting $t\to\infty$ and using (2.1), we conclude $\lim_{t\to\infty}x(t)=-\infty$. This contradicts the fact that $x(t)>0$ for $t\in[t_1,\infty)$. Thus, we have $x^{\Delta}(t)>0$ for $t\in[t_1,\infty)$. Therefore, by Lemma 1.1 we obtain

$(x^\beta(t))^\Delta = \beta x^\Delta(t)\int_0^1 [x(t)+h\mu(t)x^\Delta(t)]^{\beta-1}dh > 0$ for $t\in[t_1,\infty)$. Let $w(t) := r(t)(x^\Delta(t))^\alpha \varphi(t)/x^\beta(t)$ for $t\in[t_1,\infty)$. Then by the formulas $(FG)^\Delta = F^\Delta G + F^\sigma G^\Delta$ and $\left(F/G\right)^\Delta = F^\Delta/G^\sigma - FG^\Delta/(GG^\sigma)$ for the delta derivatives of the product FG and the quotient F/G of differentiable functions F and G, where σ is the forward jump operator on T , $F^\sigma := F\circ\sigma$ and $G^\sigma := G\circ\sigma$, we get

$$w^\Delta = \left(r(x^\Delta)^\alpha\right)^\Delta \varphi/x^\beta + \left(r(x^\Delta)^\alpha\right)^\sigma \left(\varphi/x^\beta\right)^\Delta$$
$$= \left(r(x^\Delta)^\alpha\right)^\Delta \varphi/x^\beta + \left(r(x^\Delta)^\alpha\right)^\sigma \left[\varphi^\Delta/(x^\beta)^\sigma - \varphi(x^\beta)^\Delta/x^\beta(x^\beta)^\sigma\right].$$

For $t\ge t_1$, since $(x^\beta(t))^\Delta>0$, $t\le\sigma(t)$, $x^\Delta(t)>0$ and $(r(t)(x^\Delta(t))^\alpha)^\Delta = -q(t)x^\beta(t)<0$, we have

$$w^\Delta < \left(r(x^\Delta)^\alpha\right)^\Delta \varphi/x^\beta + \left(r(x^\Delta)^\alpha\right)^\sigma \varphi^\Delta/(x^\beta)^\sigma = -q\varphi + \left(r(x^\Delta)^\alpha\right)^\sigma \varphi^\Delta/(x^\beta)^\sigma$$
$$\le -q\varphi + \varphi^\Delta r(x^\Delta)^\alpha/x^\beta . \qquad (2.3)$$

Since

$$x(t) = x(t_1) + \int_{t_1}^t x^\Delta(s)\Delta s = x(t_1) + \int_{t_1}^t r^{-1/\alpha}(s)\left(r(s)(x^\Delta(s))^\alpha\right)^{1/\alpha}\Delta s$$
$$\ge r^{1/\alpha}(t)x^\Delta(t)\int_{t_1}^t r^{-1/\alpha}(s)\Delta s \ \text{ for } t\ge t_1,$$

we find $\left(x^\Delta(t)/x(t)\right)^\alpha \le r^{-1}(t)\left(\int_{t_1}^t r^{-1/\alpha}(s)\Delta s\right)^{-\alpha} := u^\alpha(t)/r(t)$ for $t>t_1$. Thus, from (2.3) we obtain

$$w^\Delta < -q\varphi + \varphi^\Delta u^\alpha x^{\alpha-\beta} \ \text{ on } (t_1,\infty). \qquad (2.4)$$

Next, we consider the following three cases:

Case (i). Let $\alpha<\beta$. For $t\in[t_1,\infty)$, since $x(t)\ge x(t_1)>0$, we have

$$x^{\alpha-\beta}(t) \le (x(t_1))^{\alpha-\beta} := c_1 . \qquad (2.5)$$

Case (ii). Let $\alpha=\beta$. Then, for $t\in[t_1,\infty)$ we get

$$x^{\alpha-\beta}(t) = 1 . \qquad (2.6)$$

Case (iii). Let $\alpha>\beta$. Since $r(t)(x^\Delta(t))^\alpha \le r(t_1)(x^\Delta(t_1))^\alpha := b$ for $t\in[t_1,\infty)$, we obtain $x^\Delta(t)\le b^{1/\alpha}r^{-1/\alpha}(t)$ for $t\in[t_1,\infty)$. Integrating both sides of the last inequality from t_1 to t, we have $x(t)\le x(t_1)+b^{1/\alpha}\int_{t_1}^t r^{-1/\alpha}(s)\Delta s$ for $t\in[t_1,\infty)$. Therefore, there exist a constant $b_1>0$ and $t_4>t_1$ such that $x(t)\le b_1\int_{t_1}^t r^{-1/\alpha}(s)\Delta s := b_1u^{-1}(t)$ for $t\in[t_4,\infty)$. Hence, for $t\in[t_4,\infty)$ we get

$$x^{\alpha-\beta}(t) \le b_1^{\alpha-\beta}u^{\beta-\alpha}(t) := c_2u^{\beta-\alpha}(t) . \qquad (2.7)$$

Thus, for $t \geq t_4$, from (2.4)-(2.7) it follows that $w^{\Delta}(t) < -q(t)\varphi(t) + \varphi^{\Delta}(t)u^{\alpha}(t)v(t)$. Integrating both sides of the last inequality from t_4 to t, we obtain $\int_{t_4}^{t}\left[q(s)\varphi(s) - \varphi^{\Delta}(s)u^{\alpha}(s)v(s)\right]\Delta s \leq w(t_4) - w(t) < w(t_4)$ for $t \geq t_4$. Hence, we get $\limsup_{t\to\infty}\int_{t_4}^{t}\left[q(s)\varphi(s) - \varphi^{\Delta}(s)u^{\alpha}(s)v(s)\right]\Delta s \leq w(t_4) < \infty$, which contradicts (2.2). The proof is complete.

Theorem 2.2. *Suppose that (H*$_1$*), (H*$_2$*) and* $\int_{t_0}^{\infty} r^{-1/\alpha}(t)\Delta t < \infty$ *hold. Furthermore, assume that there exists a positive nondecreasing delta differentiable function* φ *such that for all* $T > t_1 \geq t_0$, *(2.2) and the following condition hold:*

$$\int_{T}^{\infty}\left(r^{-1}(z)\int_{T}^{z}\psi^{\beta}(s)q(s)\Delta s\right)^{1/\alpha}\Delta z = \infty, \tag{2.8}$$

where $\psi(t) := \int_{t}^{\infty} r^{-1/\alpha}(s)\Delta s$. *Then (1.1) is oscillatory.*

Proof. Assume that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \in [t_1, \infty)$. Therefore, there are two cases for the sign of $x^{\Delta}(t)$. The proof when $x^{\Delta}(t)$ is eventually positive is similar to that of Theorem 2.1 and hence is omitted.

Next, assume that $x^{\Delta}(t)$ is eventually negative. Then there exists $t_2 \geq t_1$ such that $x^{\Delta}(t) < 0$ for $t \in [t_2, \infty)$. Thus, from (1.1) we have $\left(r(t)(-x^{\Delta}(t))^{\alpha}\right)^{\Delta} = q(t)x^{\beta}(t) > 0$ for $t \in [t_2, \infty)$, which implies that $r(t)(-x^{\Delta}(t))^{\alpha}$ is strictly increasing on $[t_2, \infty)$. Hence, we have $r(s)(-x^{\Delta}(s))^{\alpha} \geq r(t)(-x^{\Delta}(t))^{\alpha}$ for $s \geq t \geq t_2$. Then for $s \geq t \geq t_2$ we conclude $-x^{\Delta}(s) \geq r^{-1/\alpha}(s)r^{1/\alpha}(t)(-x^{\Delta}(t))$. Integrating both sides of the last inequality from $t \geq t_2$ to $z \geq t$ and letting $z \to \infty$, for $t \in [t_2, \infty)$ we get

$$\begin{aligned} x(t) &\geq \left(\int_{t}^{\infty} r^{-1/\alpha}(s)\Delta s\right) r^{1/\alpha}(t)(-x^{\Delta}(t)) \\ &:= \psi(t)r^{1/\alpha}(t)(-x^{\Delta}(t)) \geq \psi(t)r^{1/\alpha}(t_2)(-x^{\Delta}(t_2)) := c\psi(t), \end{aligned}$$

where $c := -r^{1/\alpha}(t_2)x^{\Delta}(t_2) > 0$. Thus, from (1.1) we obtain $\left(r(t)(-x^{\Delta}(t))^{\alpha}\right)^{\Delta} = q(t)x^{\beta}(t) \geq c^{\beta}\psi^{\beta}(t)q(t)$ for $t \in [t_2, \infty)$. Integrating both sides of the last inequality from t_2 to t, for $t \in [t_2, \infty)$ we have

$$r(t)(-x^{\Delta}(t))^{\alpha} \geq r(t_2)(-x^{\Delta}(t_2))^{\alpha} + c^{\beta}\int_{t_2}^{t}\psi^{\beta}(s)q(s)\Delta s > c^{\beta}\int_{t_2}^{t}\psi^{\beta}(s)q(s)\Delta s.$$

Hence, we obtain $-x^{\Delta}(t) > \left(r^{-1}(t)c^{\beta}\int_{t_2}^{t}\psi^{\beta}(s)q(s)\Delta s\right)^{1/\alpha}$ for $t \in [t_2, \infty)$. Integrating both sides of the last inequality from t_2 to t, we find $x(t) \le x(t_2) - c^{\beta/\alpha}\int_{t_2}^{t}\left(r^{-1}(z)\int_{t_2}^{z}\psi^{\beta}(s)q(s)\Delta s\right)^{1/\alpha}\Delta z$ for $t \in [t_2, \infty)$. Letting $t \to \infty$ and using (2.8), we see $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the fact that $x(t) > 0$ for $t \in [t_1, \infty)$. The proof is complete.

Acknowledgements

This work was supported by the Natural Science Foundation of Hunan Province of P. R. China (Grant No. 11JJ3010).

References

[1] A. K. Tripathy, Some ocillation results for second order nonlinear dynamic equations of neutral type, Nonlinear Anal., **71** (2009), 1727-1735.

[2] D.-X. Chen, Oscillation of second-order Emden–Fowler neutral delay dynamic equations on time scales, Math. Comput. Modelling, **51** (2010), 1221-1229.

[3] D.-X. Chen, Oscillation and asymptotic behavior for nth-order nonlinear neutral delay dynamic equations on time scales, Acta Appl. Math., **109** (2010), 703-719.

[4] L. Erbe, J. Baoguo and A. Peterson, Oscillation of nth order superlinear dynamic equations on time scales, Rocky Mountain J. Math., **41** (2011), 471-491.

[5] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Application, Birkhäuser, Boston, 2001.

[6] Q. Yang and Z. Xu, Oscillation criteria for second order quasilinear neutral delay differential equations on time scales, Comput. Math. Appl., **62** (2011), 3682-3691.

[7] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, Results Math., **18** (1990), 18-56.

[8] S.H. Saker, Oscillation criteria of second-order half-linear dynamic equations on time scales, J. Comput. Appl. Math., **177** (2005), 375-387.

[9] S.R. Grace, R.P. Agarwal, B. Kaymakçalan and W. Sae-Jie, On the oscillation of certain second order nonlinear dynamic equations, Math. Comput. Modelling, **50** (2009), 273-286.

[10] T.S. Hassan, Oscillation criteria for half-linear dynamic equations on time scales, J. Math. Anal. Appl., **345** (2008), 176-185.

Received: December, 2011