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Clifford Windowed Fourier Transform Applied to Linear Time-Varying Systems

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Abstract

We start with an introduction about the Clifford windowed Fourier transform (CWFT). We then apply the CWFT to a linear time-varying system.

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1 Introduction

Several attempts have been made recently to generalize the classical windowed Fourier transform (WFT) to quaternion algebra. The first generalization is started by Bülow and Sommer [1, 4] who introduced a special case of the quaternionic windowed Fourier transform (QWFT) known as the quaternionic Gabor filters. They applied these filters to obtain a local two-dimensional quaternionic phase. Their generalization is obtained using the inverse (two-sided) quaternion Fourier kernel.

The WFT has been studied in quaternion algebra framework [6, 8]. It is shown that many WFT properties still hold but others have to be modified. Another generalization using the kernel of the $Cl_{n,0}$ Clifford Fourier transform [10] was introduced in [5, 7]. In present paper, using the translation, modulation and time-frequency shifts on Clifford algebra $Cl_{n,0}$ we define the Clifford windowed Fourier transform (CWFT). We also discuss an application of the CWFT to a linear time-varying system.

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \cdots, \boldsymbol{e}_n\}$ be an orthonormal vector basis of n-dimensional Euclidean vector space $\mathbb{R}^n$. The real Clifford algebra over $\mathbb{R}^n$ denoted by $Cl_{n,0}$ then has the graded 2^n-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \cdots, i_n = \boldsymbol{e}_1\boldsymbol{e}_2\cdots\boldsymbol{e}_n\}. \tag{1}$$

Obviously, for $n = 2$ (mod 4) the pseudoscalar $i_n = \boldsymbol{e}_1\boldsymbol{e}_2\cdots\boldsymbol{e}_n$ anti-commutes with each basis of Clifford algebra while i_n^2 = -1. The associative geometric multiplication of the basis vectors is governed by the rules:

$$\begin{aligned} \boldsymbol{e}_k\,\boldsymbol{e}_l &= -\boldsymbol{e}_l\,\boldsymbol{e}_k \qquad \text{for} \quad k \neq l, \qquad 1 \leq k, l \leq n, \\ \boldsymbol{e}_k^2 &= 1 \qquad\qquad \text{for} \qquad 1 \leq k \leq n. \end{aligned} \tag{2}$$

An element of Clifford algebra are called a multivector and has the following form

$$f = \sum_A \boldsymbol{e}_A f_A, \tag{3}$$

where $f_A \in \mathbb{R}$, $\boldsymbol{e}_A = \boldsymbol{e}_{\alpha_1\alpha_2\cdots\alpha_k} = \boldsymbol{e}_{\alpha_1}\boldsymbol{e}_{\alpha_2}\cdots\boldsymbol{e}_{\alpha_k}$, and $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq n$ with $\alpha_j \in \{1, 2, \cdots n\}$. For convenience, we introduce $\langle f \rangle_k = \sum_{|A|=k} f_A \boldsymbol{e}_A$ to denote *k-vector part* of f $(k = 0, 1, 2, \cdots, n)$, then

$$f = \sum_{k=0}^{k=n} \langle f \rangle_k = \langle f \rangle + \langle f \rangle_1 + \langle f \rangle_2 + \cdots + \langle f \rangle_n, \tag{4}$$

where $\langle \ldots \rangle_0 = \langle \ldots \rangle$.

A multivector $f \in Cl_{n,0}, n = 2$ (mod 4) may be decomposed as a sum of even part f_{even} and odd part f_{odd}. Thus we have

$$f = f_{even} \oplus f_{odd}, \tag{5}$$

where

$$\begin{aligned} f_{even} &= \langle f \rangle + \langle f \rangle_2 + \cdots + \langle f \rangle_r, \qquad r = 2s, s \in \mathbb{N}, s \leq \frac{n}{2}, \\ f_{odd} &= \langle f \rangle_1 + \langle f \rangle_3 + \cdots + \langle f \rangle_r, \qquad r = 2s+1, s \in \mathbb{N}, s < \frac{n}{2}. \end{aligned} \tag{6}$$

The reverse $\tilde{f}$ of a multivector f is an anti-automorphism given by

$$\widetilde{f} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle f \rangle_k, \tag{7}$$

and hence

$$\widetilde{fg} = \tilde{g}\tilde{f} \quad \text{for arbitrary } f, g \in Cl_{n,0}. \tag{8}$$

For two multivector functions $f, g \in Cl_{n,0}$, an inner product is defined by

$$(f, g)_{L^2(\mathbb{R}^n; Cl_{n,0})} = \int_{\mathbb{R}^n} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A \widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^n} f_A(\boldsymbol{x}) g_B(\boldsymbol{x})\, d^n\boldsymbol{x}. \tag{9}$$

In particular, if $f = g$, then the scalar part of the above inner product gives the L^2-norm

$$\|f\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})} = \int_{\mathbb{R}^n} \sum_A f_A^2(\boldsymbol{x})\, d^n\boldsymbol{x}. \tag{10}$$

In what follows, we introduce the $Cl_{n,0}$ Clifford Fourier transform (CFT). For detailed discussions of the properties of the CFT and their proofs, see e.g. [2, 10].

Definition 1.1 *The CFT of* $f \in L^1(\mathbb{R}^n; Cl_{n,0})$ *is the function* $\mathcal{F}\{f\}\colon \mathbb{R}^n \to Cl_{n,0}$, $n = 2 \pmod 4$ *given by*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}, \tag{11}$$

with $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^n$.

Theorem 1.2 *Suppose that* $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ *and* $\mathcal{F}\{f\} \in L^1(\mathbb{R}^n; Cl_{n,0})$. *Then the CFT of* $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ *is invertible and its inverse is calculated by*

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega}. \tag{12}$$

As in the classical case, we obtain Plancherel's formula for the CFT (see [10])

$$(f, g)_{L^2(\mathbb{R}^n;Cl_{n,0})} = \frac{1}{(2\pi)^n} (\mathcal{F}\{f\}, \mathcal{F}\{g\})_{L^2(\mathbb{R}^n;Cl_{n,0})}. \tag{13}$$

In particular, with $f = g$, taking scalar part of (13) we get scalar Parseval's formula,

$$\|f\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})} = \frac{1}{(2\pi)^n} \|\mathcal{F}\{f\}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}. \tag{14}$$

For the sake of simplicity, if not otherwise stated, n is assumed to be $n = 2 \pmod 4$ in the rest of this section.

2 Translation and Modulation in Clifford Algebra $Cl_{n,0}$

Before we define the Clifford windowed Fourier transform (CWFT) we need to introduce some notations. For a multivector function $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ we define the translation, modulation and time-frequency shifts as follows:

$$T_{\boldsymbol{b}} f(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{b}), \quad M_{\boldsymbol{\omega}} f(\boldsymbol{x}) = e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}) \tag{15}$$

and

$$M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} f(\boldsymbol{x}) = e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}-\boldsymbol{b}), \quad \boldsymbol{b}, \boldsymbol{\omega} \in \mathbb{R}^n. \tag{16}$$

Just as the classical case, we obtain the canonical commutation relations (see [9])

$$T_{\boldsymbol{b}} M_{\boldsymbol{\omega}} f = e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{b}} M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} f. \tag{17}$$

The following lemma describes the behavior of time-frequency shifts under the CFT.

Lemma 2.1 *For $\boldsymbol{x}$, $\boldsymbol{b} \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ we have*

$$\mathcal{F}\{M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} f\} = M_{\boldsymbol{b}} T_{-\boldsymbol{\omega}} \mathcal{F}\{f_{odd}\}\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{b}} + M_{-\boldsymbol{b}} T_{\boldsymbol{\omega}} \mathcal{F}\{f_{even}\}\, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{b}}. \tag{18}$$

Proof. A simple computation gives

$$\begin{aligned}
&\mathcal{F}\{M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} f\}(\boldsymbol{\omega}') \\
&= \int_{\mathbb{R}^n} e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} f(\boldsymbol{x}-\boldsymbol{b})\, e^{-i_n \boldsymbol{\omega}'\cdot\boldsymbol{x}} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} (f_{odd}(\boldsymbol{x}-\boldsymbol{b}) + f_{even}(\boldsymbol{x}-\boldsymbol{b}))\, e^{-i_n \boldsymbol{\omega}'\cdot\boldsymbol{x}} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} e^{i_n \boldsymbol{\omega}'\cdot\boldsymbol{x}} f_{odd}(\boldsymbol{x}-\boldsymbol{b})\, d^n\boldsymbol{x} + \int_{\mathbb{R}^n} e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, e^{-i_n \boldsymbol{\omega}'\cdot\boldsymbol{x}} f_{even}(\boldsymbol{x}-\boldsymbol{b})\, d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} e^{-i_n(-\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{x}} f_{odd}(\boldsymbol{x}-\boldsymbol{b})\, d^n\boldsymbol{x} + \int_{\mathbb{R}^n} e^{-i_n(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{x}} f_{even}(\boldsymbol{x}-\boldsymbol{b})\, d^n\boldsymbol{x}
\end{aligned} \tag{19}$$

Making the substitution $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{b}$ in the above expression we immediately obtain

$$\begin{aligned}
&\mathcal{F}\{M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} f\}(\boldsymbol{\omega}') \\
&= \int_{\mathbb{R}^n} e^{-i_n(-\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot(\boldsymbol{y}+\boldsymbol{b})} f_{odd}(\boldsymbol{y})\, d^n\boldsymbol{y} + \int_{\mathbb{R}^n} e^{-i_n(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot(\boldsymbol{y}+\boldsymbol{b})} f_{even}(\boldsymbol{y})\, d^n\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} e^{-i_n(-\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{b}} f_{odd}(\boldsymbol{y})\, e^{-i_n(\boldsymbol{\omega}'+\boldsymbol{\omega})\cdot\boldsymbol{y}}\, d^n\boldsymbol{y} \\
&\quad + \int_{\mathbb{R}^n} e^{-i_n(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{b}} f_{even}(\boldsymbol{y}) e^{-i_n(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{y}}\, d^n\boldsymbol{y} \\
&= e^{i_n \boldsymbol{\omega}'\cdot\boldsymbol{b}} \widehat{f_{odd}}(\boldsymbol{\omega}'+\boldsymbol{\omega})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{b}} + e^{-i_n \boldsymbol{\omega}'\cdot\boldsymbol{b}} \widehat{f_{even}}(\boldsymbol{\omega}'-\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{b}},
\end{aligned} \tag{20}$$

which was to be proved. □

Substituting the commutation relation (17) into (18), we easily obtain

$$\mathcal{F}\{M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} f\} = T_{-\boldsymbol{\omega}} M_{\boldsymbol{b}} \mathcal{F}\{f_{odd}\} + T_{\boldsymbol{\omega}} M_{-\boldsymbol{b}} \mathcal{F}\{f_{even}\}. \tag{21}$$

3 Clifford Windowed Fourier Transform

In this section we give the definition of the Clifford windowed Fourier transform (CWFT) and we then express the CWFT in the terms of the CFT.

Definition 3.1 *The CWFT of a multivector-valued signal $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ with respect to a non-zero Clifford window $\phi \in L^2(\mathbb{R}^n; Cl_{n,0})$ is given by*

$$G_\phi f(\boldsymbol{b},\boldsymbol{\omega}) = (f, M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} \phi)_{L^2(\mathbb{R}^n;Cl_{n,0})} = \int_{\mathbb{R}^n} f(\boldsymbol{x})\ \widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{x}. \quad (22)$$

It is easy to see that equation (22) can be written as

$$G_\phi f(\boldsymbol{b},\boldsymbol{\omega}) = \mathcal{F}\{f\,\tilde{\phi}(\cdot - \boldsymbol{b})\}(\boldsymbol{\omega}) = \mathcal{F}\{f\, T_{\boldsymbol{b}}\tilde{\phi}\}(\boldsymbol{\omega}). \quad (23)$$

The following result provides an alternative form of Lemma 3.5 in [7].

Theorem 3.2 *In the Clifford Fourier representation, the CWFT (22) takes the form*

$$G_\phi f(\boldsymbol{b},\boldsymbol{\omega}) = \frac{e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{b}}}{(2\pi)^n}\left[G_{\widehat{\phi}_{odd}} \hat{f}(-\boldsymbol{\omega},\boldsymbol{b}) + G_{\widehat{\phi}_{even}} \hat{f}(\boldsymbol{\omega},-\boldsymbol{b})\right]. \quad (24)$$

Proof. This can be seen from

$$\begin{aligned}
&G_\phi f(\boldsymbol{b},\boldsymbol{\omega})\\
&= (f, M_{\boldsymbol{\omega}} T_{\boldsymbol{b}} \phi)_{L^2(\mathbb{R}^n;Cl_{n,0})}\\
&\overset{(13)}{=} \frac{1}{(2\pi)^n}\,(\widehat{f}, \widehat{M_{\boldsymbol{\omega}} T_{\boldsymbol{b}}}\phi)_{L^2(\mathbb{R}^n;Cl_{n,0})}\\
&\overset{(18)}{=} \frac{1}{(2\pi)^n}\,(\widehat{f}, M_{\boldsymbol{b}} T_{-\boldsymbol{\omega}} \mathcal{F}\{\phi_{odd}\}\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} + M_{-\boldsymbol{b}} T_{\boldsymbol{\omega}} \mathcal{F}\{\phi_{even}\}\, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\\
&= \frac{1}{(2\pi)^n}(\widehat{f}, M_{\boldsymbol{b}} T_{-\boldsymbol{\omega}} \mathcal{F}\{\phi_{odd}\}\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\\
&\quad + \frac{1}{(2\pi)^n}(\widehat{f}, M_{-\boldsymbol{b}} T_{\boldsymbol{\omega}} \mathcal{F}\{\phi_{even}\}\, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\\
&= \frac{1}{(2\pi)^n}(\widehat{f}, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} M_{\boldsymbol{b}} T_{-\boldsymbol{\omega}} \mathcal{F}\{\phi_{odd}\})_{L^2(\mathbb{R}^n;Cl_{n,0})}\\
&\quad + \frac{1}{(2\pi)^n}(\widehat{f}, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} M_{-\boldsymbol{b}} T_{\boldsymbol{\omega}} \mathcal{F}\{\phi_{even}\})_{L^2(\mathbb{R}^n;Cl_{n,0}}\\
&= \frac{1}{(2\pi)^n}(\widehat{f}, M_{\boldsymbol{b}} T_{-\boldsymbol{\omega}} \mathcal{F}\{\phi_{odd}\})_{L^2(\mathbb{R}^n;Cl_{n,0})}\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\\
&\quad + \frac{1}{(2\pi)^n}(\widehat{f}, M_{-\boldsymbol{b}} T_{\boldsymbol{\omega}} \mathcal{F}\{\phi_{even}\})_{L^2(\mathbb{R}^n;Cl_{n,0})}\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\\
&= \frac{e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}}{(2\pi)^n}\left[G_{\widehat{\phi_{odd}}} \hat{f}(-\boldsymbol{\omega},\boldsymbol{b}) + G_{\widehat{\phi_{even}}} \hat{f}(\boldsymbol{\omega},-\boldsymbol{b})\right], \qquad (25)
\end{aligned}$$

which gives the desired result. □

4 Application of CWFT

This section discusses an application of the CWFT to study n-dimensional linear TV systems. We may regard the CWFT as a linear TV band-pass filter element of a filter-bank spectrum analyzer and, therefore, the TV spectrum obtained by the CWFT can also be interpreted as the output of such a linear TV band-pass filter element.

Definition 4.1 *Consider an n-dimensional linear TV system with $h(\cdot,\cdot,\cdot)$ denoting the Clifford impulse response of the filter. The output $r(\cdot,\cdot)$ of the linear TV system is defined by*

$$r(\boldsymbol{b},\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) h(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{b}-\boldsymbol{x})\, d^n\boldsymbol{x} = \int_{\mathbb{R}^n} f(\boldsymbol{b}-\boldsymbol{x}) h(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{x})\, d^n\boldsymbol{x}, \tag{26}$$

where $f(\cdot)$ is an n-dimensional Clifford valued input signal.

We then obtain the transfer function $R(\cdot,\cdot)$ of the Clifford impulse response $h(\cdot,\cdot,\cdot)$ of the TV filter as

$$R(\boldsymbol{b},\boldsymbol{\omega}) = \int_{\mathbb{R}^n} h(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{\alpha}} d^n\boldsymbol{\alpha}, \quad \boldsymbol{\alpha}\in\mathbb{R}^n. \tag{27}$$

The following simple theorem (compare to [6]) relates the CWFT to the output of a linear TV band-pass filter.

Theorem 4.2 *Consider a linear TV band-pass filter. Let the TV Clifford impulse response $h_1(\cdot,\cdot,\cdot)$ of the filter be defined by*

$$h_1(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha}) = e^{-i_n \boldsymbol{\omega}\cdot(\boldsymbol{b}-\boldsymbol{\alpha})}\, \widetilde{\phi(-\boldsymbol{\alpha})}, \tag{28}$$

where $\phi(\cdot)$ is the Clifford window function. The output $r_1(\cdot,\cdot)$ of the TV system is $G_{\phi_{even}} f(\boldsymbol{b},\boldsymbol{\omega}) + G_{\phi_{odd}} f(\boldsymbol{b},-\boldsymbol{\omega})$.

Proof. Using Definition 4.1, we get the output as follows:

$$\begin{aligned} r_1(\boldsymbol{b},\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, h_1(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{b}-\boldsymbol{x})\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot(\boldsymbol{b}-(\boldsymbol{b}-\boldsymbol{x}))}\, \widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, \widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\, \boldsymbol{x}}\, \left(\{\phi_{even}(\boldsymbol{x}-\boldsymbol{b})\}^{\sim} + \{\phi_{odd}(\boldsymbol{x}-\boldsymbol{b})\}^{\sim}\right)\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x}) \{\phi_{even}(\boldsymbol{x}-\boldsymbol{b}\}^{\sim}\, e^{-i_n \boldsymbol{\omega}\cdot\, \boldsymbol{x}}\, d^n\boldsymbol{x} \\ &\quad + \int_{\mathbb{R}^n} f(\boldsymbol{x}) \{\phi_{odd}(\boldsymbol{x}-\boldsymbol{b}\}^{\sim}\, e^{i_n \boldsymbol{\omega}\cdot\, \boldsymbol{x}}\, d^n\boldsymbol{x} \\ &= G_{\phi_{even}} f(\boldsymbol{b},\boldsymbol{\omega}) + G_{\phi_{odd}} f(\boldsymbol{b},-\boldsymbol{\omega}), \end{aligned} \tag{29}$$

which proves the theorem. □

Remark 4.1 *Notice that if the Clifford impulse response in (28) is defined by*

$$h_1(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha}) = \widetilde{\phi(-\boldsymbol{\alpha})}\, e^{-i_n\boldsymbol{\omega}\cdot(\boldsymbol{b}-\boldsymbol{\alpha})} \tag{30}$$

then the output $r_1(\cdot,\cdot)$ *of the TV system is* $G_\phi f(\boldsymbol{b},\boldsymbol{\omega})$.

We see that the choice of the Clifford impulse response of the filter will determine a characteristic output of the linear TV systems. For example, if we translate the TV Clifford impulse response $h_1(\cdot,\cdot,\cdot)$ by $\boldsymbol{b}_0 = b_{01}\boldsymbol{e}_1 + b_{02}\boldsymbol{e}_2 + \cdots + b_{0n}\boldsymbol{e}_n$, i.e.

$$h_1(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha}) \to h_1(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha}-\boldsymbol{b}_0) = \widetilde{\phi(-(\boldsymbol{\alpha}-\boldsymbol{b}_0))}\, e^{-i_n\boldsymbol{\omega}\cdot(\boldsymbol{b}-(\boldsymbol{\alpha}-\boldsymbol{b}_0))} \tag{31}$$

then the output is according to the delay property of the CFT

$$r_{1,\boldsymbol{b}_0}(\boldsymbol{b},\boldsymbol{\omega}) = G_\phi f(\boldsymbol{b}-\boldsymbol{b}_0,\boldsymbol{\omega})\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}_0}. \tag{32}$$

Remark 4.2 *It is not difficult to see that if the Clifford impulse response in (31) takes the form*

$$h_1(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha}) \to h_1(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha}-\boldsymbol{b}_0) = e^{-i_n\boldsymbol{\omega}\cdot(\boldsymbol{b}-(\boldsymbol{\alpha}-\boldsymbol{b}_0))}\widetilde{\phi(-(\boldsymbol{\alpha}-\boldsymbol{b}_0))}, \tag{33}$$

then its output is

$$r_{1,\boldsymbol{b}_0}(\boldsymbol{b},\boldsymbol{\omega}) = G_{\phi_{even}} f(\boldsymbol{b}-\boldsymbol{b}_0,\boldsymbol{\omega})\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}_0} + G_{\phi_{odd}} f(\boldsymbol{b}-\boldsymbol{b}_0,-\boldsymbol{\omega})\, e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{b}_0}. \tag{34}$$

Theorem 4.3 *Consider a linear TV band-pass filter with the TV Clifford impulse response* $h_2(\cdot,\cdot,\cdot)$ *defined by*

$$h_2(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{\alpha}) = e^{-i_n\boldsymbol{\omega}\cdot(\boldsymbol{b}-\boldsymbol{\alpha})}. \tag{35}$$

If the input to this system is the Clifford signal $f(\boldsymbol{x})$*, its output* $r_2(\boldsymbol{\omega}) = r_2(\boldsymbol{\omega},\cdot)$ *is, independent of the* $\boldsymbol{b}$*-argument, equal to the CFT of* f*:*

$$r_2(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega}). \tag{36}$$

Proof. Using Definition 4.1, we obtain

$$\begin{aligned} r_2(\boldsymbol{b},\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, h_2(\boldsymbol{b},\boldsymbol{\omega},\boldsymbol{b}-\boldsymbol{x})\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n\boldsymbol{\omega}\cdot(\boldsymbol{b}-(\boldsymbol{b}-\boldsymbol{x}))}\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \\ &= \mathcal{F}\{f\}(\boldsymbol{\omega}). \end{aligned} \tag{37}$$

Or $r_2(\boldsymbol{\omega}) = r_2(\boldsymbol{b},\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega})$, because the right-hand side of (37) is independent of $\boldsymbol{b}$. □

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