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# Dynamic DEA: Relative Efficiency Measure of Units

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**Abstract**

Dynamic data envelopment analysis (DDEA) is an approach which deals with assessing performance a group of DMUs (decision making units) during several periods of time. Actually, what discriminates between DDEA and other existing techniques depending on time factor such as Window analysis and the Malmquist index, is the attention to carry-over or investment activities and actually one of the most important feature of DDEA is this fact. This paper develops some assumptions into DDEA which can give managers new insights into what does happen to those investments during periods of an assessment window and how they will be effective in the future. In continue, some factors that affect on those investments within different periods are studied. Consequently, using the new insight we develop the maximum dynamic relative efficiency measure (DRE) of a DMU under evaluation over whole periods of an assessment window.

**Mathematics Subject Classification:** 90

**Keywords:** DEA; DDEA; DRE; Investment activity

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# 1 Introduction

Data envelopment analysis (DEA) pioneered by Charnes et al. [1] is an optimization tool based on mathematical programming generalizing the Farrell [5] single-output/multiple-input technical efficiency measure under constant returns to scale (TE-CRS) to multiple-output/multiple-input case. After that, dynamic DEA (DDEA, hereafter) was originally developed by Fare et al. [4] to cope with long time assessment incorporating concepts of quasi-fixed inputs and/or investment activities. DDEA enables us to survey the performance of a DMU under evaluation regarding the effect of carry-over or investment activities in a period on the DMU's activity in another period. This feature of DDEA discriminates it from the separate time models such as Window analysis [6] and Malmquist productivity index [3]. So far, many DDEA models have been developed in this area, e.g. Nemoto and Goto [8,9], Sueyoshi and Sekitani [10]. In those DDEA research efforts, several concepts have been extended into DDEA context such as returns to scale (RTS), the cost minimization model, Nevertheless, relative efficiency measure of DMUs over an assessment window has not been mentioned yet. Therefore, we refer to extend this measure into DDEA and then propose dynamic relative efficiency measure. As can be seen in the literature of DEA, the TDT measure [2] is used as a relative efficiency measure (RE) of a $DMU_p$ introduced by the "ratio of ratios" model as follows:

$$\begin{aligned} &\max_{u,v} \quad \frac{\sum_{r=1}^{s} u_r y_{r,p}}{\sum_{i=1}^{m} v_i x_{i,p}} \Big/ \frac{\sum_{r=1}^{s} u_r y_{r,k}}{\sum_{i=1} v_i x_{i,k}} \\ &\text{where} \quad \frac{\sum_{r=1}^{s} u_r y_{r,k}}{\sum_{i=1}^{m} v_i x_{i,k}} = \text{maximum}_{j=1,2,\ldots,n} \left\{ \frac{\sum_{r=1}^{s} u_r y_{r,j}}{\sum_{i=1}^{m} v_i x_{i,j}} \right\}. \end{aligned} \tag{1}$$

Actually, the model (1) is a maximin DEA model that can be seen as a non-normalized model maximizing the relative efficiency of a $DMU_p$. It must be noted that to extent the concept of the relative efficiency measure into DDEA, we firstly develop some assumptions related to investment activities into the DDEA that are relatively different from the those mentioned by previous researchers such as Nemoto and Goto (NG hereafter) and other ones. Therefore, firstly we survey DDEA framework from NG viewpoint and then suggest some assumptions into DDEA related to investment activities in order to develop a new version of DDEA. The remainder of this paper unfolds into four sections: In section 2, we review some interpretations related to DDEA models. In section 3, firstly we introduce a new version of DDEA framework due to investment activities and then consequently propose dynamic relative efficiency measure over an assessment window. Finally, the paper will be ended by some brief concluding comments in section 4.

# 2 The DDEA

In this section we review some assumptions of DDEA framework from NG viewpoint. To this end, we deal with n DMUs (j=1,2,...,n) examined in T periods (t=1,2,...,T) collected into a set called assessment window, $W = \{t|t = 1, 2, ..., T\}$. In the period t, each $DMU_j$ uses two different groups of inputs: $k_{t-1,j} \in \mathbb{R}^l_+$ (as a vector of quasi-fixed inputs at the start of period t) and $x_{t,j} \in \mathbb{R}^m_+$ (as a vector of variable inputs or current inputs) to produce two different groups of outputs: $y_{t,j} \in \mathbb{R}^s_+$ (as a vector of goods) and $k_{t,j} \in \mathbb{R}^l_+$ (as a vector of quasi-fixed inputs at the start of period t+1). Then the production possibility set in the period t will be as follows:

$$\phi_t^{CRS} = \tag{2}$$
$$\left\{(x_t, k_{t-1}, y_t, k_t) \in \mathbb{R}^{m+l}_+ \times \mathbb{R}^{s+l}_+ | X_t\lambda_t \leq x_t, K_{t-1}\lambda_t \leq k_{t-1}, Y_t\lambda_t \geq y_t, K_t\lambda_t \geq k_t, \lambda_t \geq 0\right\},$$

where $\lambda_t \in \mathbb{R}^n_+$ is a vector of weights to connect the DMUs in the period t, $X_t = [x_{t,1}, x_{t,2}, ..., x_{t,n}]$, $K_{t-1} = [k_{t-1,1}, k_{t-1,2}, ..., k_{t-1,n}]$ and $Y_t = [y_{t,1}, y_{t,2}, ..., y_{t,n}]$ are as matrices of inputs, quasi-fixed inputs and outputs, respectively. Let $DMU_p$ be under evaluation which uses $(x_t, k_{t-1})$ to produce $(y_t, k_t)$ for t=1,2,...,T. Moreover, there exist some definitions related to DDEA literature [7] that we present some of them as follows:
1. Period (term): the unit of time measurement which is usually taken to be six months unless otherwise specified.
2. Appraisal (assessment) period: the predetermined number of periods that management regards the economic evaluation of each decision making unit's performance.
3. Capital investment input variables: special input variables that could be referred as constrained discretionary variables where those constraints may come from a specific investment policy. As two of those constraints we refer to $K_t$ as fixed assets at the start of period t and $I_t$ as fixed capital investment at period t.
4. Quasi-fixed input variables: quasi-fixed or non-discretionary variables are those that may not be restricted such as the acres of land in a farm.

# 3 Main idea

## 3.1 New version of DDEA

Suppose that $\mathbf{X}_{t,p}$ is a vector of whole inputs consumed by a $DMU_p$ at the beginning of the time period t that $\mathbf{X}_{t,p} = \begin{pmatrix} x_{t,p} \\ A_{t,p}k_{t-1,p} \\ \overline{x}_p \end{pmatrix}$; where $x_{t,p} = (x_{t,p,1}, ..., x_{t,p,m})$ is a m-vector of inputs whose the components might be different from one time period to another one;

$k_{t-1,p} = (k_{t-1,p,1}, ..., k_{t-1,p,\overline{s}})$ is a $\overline{s}$-vector of the investment inputs which belong to time period t-1 for using during time period t; $A_t$ is a $s \times s$-diagonal matrix whose the $ith$ diagonal element $a_{ii}$ $(i = 1, ..., s)$ is a consumption percent of the $ith$ component of the $k_{t-1,p}$. Therefore, $A_{t,p}k_{t-1,p}$ is that portion of investment inputs vector $k_{t-1,p}$ used during time period t. The rest of the $k_{t-1,p}$ not used in time period t and sent to time period t+1 can be shown by $\widetilde{A}_{t,p}k_{t-1,p}$. The $\widetilde{A}_{t,p}$ is a $s \times s$-diagonal matrix whose $ith$ diagonal element is $1 - a_{ii}$, $(i = 1, ..., s)$ is non-consumption percent of the $k_{t-1,p,i}$ in time period t. Moreover, suppose that $\mathbf{Y}_{t,p} \in \mathbb{R}^{s+l}_+$ is as a vector of all outputs produced by the $DMU_p$ in time period t that $\mathbf{Y}_{t,p} = \begin{pmatrix} y_{t,p} \\ h_{t,p} \end{pmatrix}$. The $y_{t,p} \in \mathbb{R}^s_+$ is a vector of external outputs that are sent to markets and $h_{t,p} \in \mathbb{R}^l_+$ is a vector of internal outputs that are sent to time period t+1. Moreover, $DMU_p$ may consume a vector of quasi-fixed inputs shown by $\overline{x} \in R^{\overline{m}}_+$, whose components are identical for all DMUs in each time period such as acres of a land in a farm. To emphasize the connection of two time periods in DDEA, the following constraint must be hold for all link factors:

$$h_t + \widetilde{A}_t k_{t-1} = k_t = A_{t+1}k_t + \widetilde{A}_{t+1}k_t, \qquad t = 1, ..., T-1. \tag{3}$$

By considering the above figure, Eq. (3) could be summarized as follows:

$$h_t + \widetilde{A}_t k_{t-1} = A_{t+1}k_t + \widetilde{A}_{t+1}k_t, \qquad t = 1, ..., T-1. \tag{4}$$

## 3.2 Relative efficiency in DDEA (DRE)

As the same as before, suppose that the $DMU_p$ is examined through the assessment window W comprising T periods represented by W $= \{1, 2, ..., T\}$. Also, assume that the $DMU_p$ uses input vector $\mathbf{X}_{t,p}$ to produce output vector $\mathbf{Y}_{t,p}$. Regarding those mentioned above, we propose the Eq.(5) as an approach to develop the dynamic efficiency measure (DE) as follows:

$$\begin{aligned} DE_p = \quad & \max_{u_t,\omega_t,v_t,\mu_t} \frac{\sum_{t=1}^{T}(u_t^\tau y_{t,p} + \omega_t^\tau h_{t,p})}{\sum_{t=1}^{T}(v_t^\tau x_{t,p} + \mu_t^\tau A_t k_{t-1,p} + \sigma_t^\tau \overline{x}_p)}, \\ \text{s.t.} \quad & \frac{\sum_{t=1}^{T}(u_t^\tau y_{t,j} + \omega_t^\tau h_{t,j})}{\sum_{t=1}^{T}(v_t^\tau x_{t,j} + \mu_t^\tau A_t k_{t-1,j} + \sigma_t^\tau \overline{x}_j)} \leq 1, \quad j = 1, ..., n, \\ & u_t \in R^s_+,\ \omega_t \in R^l_+,\ \mu_t \in R^{\overline{s}}_+,\ v_t \in R^m_+,\ \sigma_t \in R^{\overline{m}}_+ \quad t = 1, ..., T. \end{aligned} \tag{5}$$

Using the Charnes-Cooper transformation makes a simpler model as a linear form like the model (6) as follows:

$$DE_p = \max_{u_t,\omega_t,v_t,\mu_t} \sum_{t=1}^{T}(u_t^\tau y_{t,p} + \omega_t^\tau h_{t,p}), \tag{6}$$

$$\text{s.t.} \quad \sum_{t=1}^{T}(v_t^\tau x_{t,p} + \mu_t^\tau A_t k_{t-1,p} + \sigma_t^\tau \overline{x}_p) = 1,$$

$$\sum_{t=1}^{T}(u_t^\tau y_{t,j} + \omega_t^\tau h_{t,j}) - \sum_{t=1}^{T}(v_t^\tau x_{t,j} + \mu_t^\tau A_{t,j} k_{t-1,j} + \sigma_t^\tau \overline{x}_j) \leq 0, \quad j = 1, ..., n,$$

$$h_{t,j} + \widetilde{A}_{t,j} k_{t-1,j} = A_{t+1,j} k_{t,j} + \widetilde{A}_{t+1,j} k_{t,j}, \quad t = 1, ..., T-1,\ j = 1, ..., n,$$

$$u_t \in R_+^s,\ \omega_t \in R_+^l,\ \mu_t \in R_+^{\overline{s}},\ v_t \in R_+^m,\ \sigma_t \in R_+^{\overline{m}} \quad t = 1, ..., T.$$

**Definition 1.** Let an optimal solution (6) to $DMU_p$ be $(u_t^*,\ v_t^*,\ \omega_t^*,\ \mu_t^*,\ \sigma_t^*)$ for (t=1,...,T). We define the dynamic efficiency measure of $DMU_p$ over the assessment window W as follows:

$$DE_p = \frac{\sum_{t=1}^{T}(u_t^{\tau *} y_{t,p} + \omega_t^{\tau *} h_{t,p})}{\sum_{t=1}^{T}(v_t^{\tau *} x_{t,p} + \mu_t^{\tau *} A_t k_{t-1,p} + \sigma_t^{\tau *} \overline{x}_j)} \tag{7}$$

Now based on the definition 1, we define the dynamic relative efficiency of the $DMU_p$ as follows:

**Definition 2.** Let an optimal solution (6) to $DMU_p$ be $(u_t^*,\ v_t^*,\ \omega_t^*,\ \mu_t^*,\ \sigma_t^*)$ for (t=1,...,T) and let the $DE_p$ be as dynamic efficiency measure of $DMU_p$, then we define its dynamic relative efficiency measure (DRE) as follows:

$$DRE_p = \frac{DE_p}{\max_{j=1,...,n} DE_j}, \tag{8}$$

$$\text{where} \quad DE_j = \frac{\sum_{t=1}^{T}(u_t^{\tau *} y_{t,j} + \omega_t^{\tau *} h_{t,j})}{\sum_{t=1}^{T}(v_t^{\tau *} x_{t,j} + \mu_t^{\tau *} A_t k_{t-1,j} + \sigma_t^{\tau *} \overline{x}_j)}.$$

**Theorem 3.1.** *The $DRE_j$ satisfies in $0 < DRE_j \leq 1$ for each $DMU_j$.*

*Proof.* Considering the Eq. (8) completes the proof. □

# 4 Conclusions

Firstly, in this paper we reviewed and paraphrased some definitions and symbols correlating with the DDEA framework from the NG'viewpoint. After that, we originally investigated the investment activity concept through an equation guaranteeing the continuity of the trace of the investment between two periods. Then consequently based on those new assumptions and definitions, we proposed an approach in which not only the efficiency measure of a $DMU_p$ was considered over the whole periods belonging to an assessment window ($DE_p$), but also the relative efficiency measure of that DMU was evaluated ($DMU_p$).

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