

Fuzzy Stability of Cubic Functional Equations: A Fixed Point Alternative Approach

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Abstract. In [6] Shin et al. proved the Hyers-Ulam-Rassias stability of the cubic functional equation

$$3f(x \pm 3y) = 15f(x \pm y) + 80f(y)$$

in NAN-spaces.

In this paper, using the fixed point method, we prove the Hyers-Ulam-Rassias stability of the above functional equation in fuzzy Banach spaces.

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1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?* If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [8] in 1940. In the next year, Hyers [1] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [5] proved a generalization of Hyers' theorem for additive mappings.

The reader is referred to ([6]-[8]) for detailed information on stability of functional equations.

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x, t) = 0$ for $t \leq 0$;

(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

(N4) $N(x + y, c + t) \geq \min\{N(x, s), N(y, t)\}$;

(N5) $N(x, \cdot)$ is a non-decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$;

(N6) for $x \neq 0$, $N(x, \cdot)$ is continuous on \mathbb{R} .

The pair (X, N) is called a fuzzy normed vector space (briefly, FNS).

Definition 1.2. Let (X, N) be an FNS. A sequence $\{x_n\}$ in X is said to converge or be convergent if there exists an $x \in X$ such that $\lim_{t \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t \rightarrow \infty} x_n = x$.

Definition 1.3. Let (X, N) be an FNS. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in an FNS is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \rightarrow Y$ between FNS X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \rightarrow Y$ is continuous at each $x \in X$, then $f : X \rightarrow Y$ is said to be continuous on X .

Theorem 1.4. Let (X, d) be a complete generalized metric space and $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that

- (a) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- (b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (c) y^* is the unique fixed point of J in the set

$$Y = \{y \in X : d(J^{n_0} x, y) < \infty\};$$

- (d) $d(y, y^*) \leq \frac{d(y, Jy)}{1-L}$ for all $y \in Y$.

In this paper, using the fixed point and direct methods, we prove the Hyers-Ulam-Rassias stability of the following reciprocal difference functional equation:

$$(1.1) \quad 3f(x \pm 3y) = 15f(x \pm y) + 80f(y)$$

in fuzzy normed spaces.

2. Fuzzy Hyers-Ulam-Rassias stability of functional equation (1.1)

Throughout this subsection, using the fixed point alternative approach we prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. In this subsection, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 2.1. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with $\varphi\left(\frac{x}{3}, \frac{y}{3}\right) \leq \frac{L}{27}\varphi(x, y)$ for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping

satisfying

$$(2.1) \quad N(3f(x \pm 3y) - 15f(x \pm y) - 80f(y), t) \geq \frac{t}{t + \varphi(x, y)}$$

for all $x, y \in X$ and all $t > 0$. Then the limit

$$C(x) := N\text{-}\lim_{n \rightarrow \infty} 27^n f\left(\frac{x}{3^n}\right)$$

exists for each $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$(2.2) \quad N(f(x) - C(x), t) \geq \frac{27(1-L)t}{27(1-L)t + L\varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = 0$ and then replacing x by $\frac{x}{3}$ in (2.1), we have

$$(2.3) \quad N\left(f(x) - 27f\left(\frac{x}{3}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{3}, 0\right)}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \rightarrow Y\}$ and the generalized metric d in S defined by

$$d(f, g) := \inf \left\{ \mu \in \mathbb{R}^+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)} \right. \\ \left. \text{for all } x \in X \text{ and all } t > 0 \right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [4, Lemma 2.1]). Now, we consider a linear mapping $J : S \rightarrow S$ such that $Jg(x) := 27g\left(\frac{x}{3}\right)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then $N(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x, 0)}$ for all $x \in X$ and $t > 0$. Hence

$$N(Jg(x) - Jh(x), L\epsilon t) = N\left(27g\left(\frac{x}{3}\right) - 27h\left(\frac{x}{3}\right), L\epsilon t\right) \\ = N\left(g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right), \frac{L\epsilon t}{27}\right) \geq \frac{\frac{L\epsilon t}{27}}{\frac{L\epsilon t}{27} + \varphi\left(\frac{x}{3}, 0\right)} \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. It follows from (2.3) that

$$N\left(f(x) - 27f\left(\frac{x}{3}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{3}, 0\right)} \geq \frac{t}{t + \frac{L}{27}\varphi(x, 0)}$$

for all $x \in X$ and $t > 0$. This means that $d(f, Jf) \leq \frac{L}{27}$. By Theorem 1.4, there exists a mapping $C : X \rightarrow Y$ satisfying the following:

(1) C is a fixed point of J , that is,

$$(2.4) \quad 27C\left(\frac{x}{3}\right) = C(x)$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that C is a unique mapping satisfying (2.4) such that there exists $\mu \in (0, \infty)$ satisfying $N(f(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, C) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$N\text{-}\lim_{n \rightarrow \infty} 27^n f\left(\frac{x}{3^n}\right) = C(x)$$

for all $x \in X$.

(3) $d(f, Q) \leq \frac{d(f, Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, C) \leq \frac{L}{27-27L}$. This implies that the inequality (2.2) holds. Furthermore, since

$$\begin{aligned} & N(3C(x \pm 3y) - 15C(x \pm y) - 80C(y), t) \\ & \geq N\text{-}\lim_{n \rightarrow \infty} N\left(3 \cdot 27^n f\left(\frac{x \pm 3y}{3^n}\right) - 15 \cdot 27^n f\left(\frac{x \pm y}{3^n}\right) - 80 \cdot 27^n f\left(\frac{y}{3^n}\right), t\right) \\ & \geq \lim_{n \rightarrow \infty} \frac{\frac{t}{27^n}}{\frac{t}{27^n} + \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}\right)} \geq \lim_{n \rightarrow \infty} \frac{\frac{t}{27^n}}{\frac{t}{27^n} + \frac{L^n}{27^n} \varphi(x, y)} \rightarrow 1 \end{aligned}$$

for all $x, y \in X, t > 0$. Thus the mapping $C : X \rightarrow Y$ is a mapping satisfying (1.1), as desired. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \rightarrow Y$ be a mapping satisfying*

$$(2.5) \quad \begin{aligned} N(3f(x \pm 3y) - 15f(x \pm y) - 80f(y), t) \\ \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then, the limit

$$C(x) := N\text{-}\lim_{n \rightarrow \infty} 27^n f\left(\frac{x}{3^n}\right)$$

exists for each $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{(3^p - 27)t}{(3^p - 27)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 3^{3-p}$ and we get the desired result. \square

Theorem 2.3. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(3x, 3y) \leq 27L\varphi(x, y)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying (2.1). Then $C(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{f(3^n x)}{27^n}$ exists for each $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$(2.6) \quad N(f(x) - C(x), t) \geq \frac{18(1-L)t}{18(1-L)t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined as in the proof of Theorem 4.1. Consider the linear mapping $J : S \rightarrow S$ such that $Jg(x) := \frac{g(3x)}{27}$ for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then $N(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x, 0)}$ for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(\frac{g(3x)}{27} - \frac{h(3x)}{27}, L\epsilon t\right) \\ &= N(g(3x) - h(3x), 27L\epsilon t) \geq \frac{27L\epsilon t}{27L\epsilon t + \varphi(3x, 0)} \geq \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. It follows from (2.3) that

$$N\left(\frac{f(3x)}{27} - f(x), \frac{t}{27}\right) \geq \frac{t}{t + \varphi(x, 0)}.$$

Therefore $d(f, Jf) \leq \frac{1}{27}$. By Theorem 1.4, there exists a mapping $C : X \rightarrow Y$ satisfying the following:

(1) C is a fixed point of J , that is,

$$(2.7) \quad 27C(x) = C(3x)$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that C is a unique mapping satisfying (2.7) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n f, C) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$N\text{-}\lim_{n \rightarrow \infty} \frac{f(3^n x)}{27^n}$$

for all $x \in X$.

(3) $d(f, C) \leq \frac{1}{1-L}d(f, Jf)$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{1}{27(1-L)}.$$

This implies that the inequality (2.6) holds. The rest of the proof is similar to that of the proof of Theorem 2.1. \square

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be a mapping satisfying (2.5). Then the limit $C(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{f(3^n x)}{27^n}$ exists for each $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that*

$$N(f(x) - C(x), t) \geq \frac{2(27 - 3^p)t}{2(27 - 3^p)t + 3\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 3^{p-3}$ and we get the desired result. \square

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