

The Nonlinear Product of the Bessel Diamond Operator and the Bessel Helmholtz Operator Related to the Bessel Biharmonic Equation

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Abstract

In this paper, we study the solution of the equation

$$\diamond_B^k(\Delta_B + m^2)^k u(x) = f(x, \Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x))$$

where $\diamond_B^k(\Delta_B + m^2)^k$ is the product of the Bessel diamond operator and the Bessel Helmholtz operator, u is an unknown generalized function, f is a generalized function, m is a positive real number and k is a nonnegative integer. It found that the existence of the solution $u(x)$ of such an equation depends on the condition of f and $\Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x)$. Moreover such a solution $u(x)$ related to the Bessel biharmonic equation depends on the conditions of p , q and k .

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1 Introduction

The operator \diamond_B^k was first introduced by Yildirim et al. [10] and is named the Bessel diamond operator iterated k times defined by

$$\diamond_B^k = \left[(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^2 - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})^2 \right]^k, \quad (1)$$

where $p + q = n$ is the dimension of the space \mathbb{R}_n^+ , $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \dots, n$. The Bessel diamond operator can be expressed as the product of the operator Δ_B and \square_B , that is $\diamond_B = \square_B \Delta_B = \Delta_B \square_B$, where Δ_B is the Laplace-Bessel operator defined by

$$\Delta_B = B_{x_1} + B_{x_2} + \dots + B_{x_n}, \tag{2}$$

and \square_B is the Bessel ultra-hyperbolic operator defined by

$$\square_B = B_{x_1} + B_{x_2} + \dots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \dots - B_{x_{p+q}}. \tag{3}$$

In this paper, we study the nonlinear equation of the form

$$\diamond_B^k (\Delta_B + m^2)^k u(x) = f(x, \Delta_B^{k-1} \square_B^k (\Delta_B + m^2)^k u(x)) \tag{4}$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of \mathbb{R}_n^+ and $\partial\Omega$ denotes the boundary of Ω . We will find the solution $u(x)$ of (4) which is unique under the condition $|f(x, \Delta_B^{k-1} \square_B^k (\Delta_B + m^2)^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\Delta_B^{k-1} \square_B^k (\Delta_B + m^2)^k u(x) = 0$ for $x \in \partial\Omega$. If we put $k = 1$, $p = n$ and $q = 0$, then we obtain a solution of the nonhomogeneous Bessel biharmonic equation.

2 Preliminary Notes

The generalized shift operator T_x^y has the following form [5],

$$T_x^y = C_v^* \int_0^\pi \dots \int_0^\pi \varphi(s_1, \dots, s_n) \left(\prod_{i=1}^n \sin^{2v_i-1} \theta_i \right) d\theta_1 \dots d\theta_n,$$

where $s_i^2 = x_i^2 + y_i^2 - 2x_i y_i \cos \theta_i$, $x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [5],

$$\begin{aligned} \frac{d^2\varphi}{dx_i^2} + \frac{2v_i}{x_i} \frac{d\varphi}{dx_i} &= \frac{d^2\varphi}{dy_i^2} + \frac{2v_i}{y_i} \frac{d\varphi}{dy_i}, \\ \varphi(x_i, 0) &= f(x), \\ \varphi_{y_i}(x_i, 0) &= 0, \end{aligned}$$

where $x_i, y_i \in \mathbb{R}_n^+$ for $i = 1, 2, \dots, n$. The convolution operator determined by the T_x^y is as follows

$$(f * \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy. \tag{5}$$

Convolution in (5) is known as a B -convolution. We note the following properties of the B -convolution and the generalized shift operator,

(a) $T_x^y \cdot 1 = 1.$

(b) $T_x^0 \cdot f(x) = f(x).$

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^n x_i^{2v_i} \right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1,$

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy.$$

(e) $(f * g)(x) = (g * f)(x).$

The Fourier-Bessel transformation and its inverse transformation are defined as follows [9],

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} \right) dy,$$

$$(F_B^{-1} f)(x) = (F_B f)(-x), \quad C_v = \left(\prod_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \right)^{-1},$$

where $J_{v_i - \frac{1}{2}}(x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. There are following equalities for Fourier-Bessel transformation [9],

$$F_B \delta(x) = 1$$

and

$$F_B(f * g)(x) = F_B f(x) \cdot F_B g(x).$$

Lemma 2.1 *Given the equation $\Delta_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where Δ_B^k is the Laplace-Bessel operator iterated k -times defined by (2). Then $u(x) = (-1)^k S_{2k}(x)$ is an elementary solution of the operator Δ_B^k , where*

$$S_{2k}(x) = \frac{2^{n+2|v|-4k} \Gamma\left(\frac{n+2|v|-2k}{2}\right)}{\prod_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \Gamma(k)} |x|^{2k-n-2|v|}. \tag{6}$$

The proof of this Lemma is given in [10].

Lemma 2.2 *Given the equation $\square_B^k u(x) = \delta(x)$ for $x \in \Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \dots, x_n > 0 \text{ and } V > 0\}$, where \square_B^k is the Bessel-ultra-hyperbolic operator iterated k -times defined by (3). Then $u(x) = R_{2k}(x)$ is an elementary solution of the operator \square_B^k , where*

$$R_{2k}(x) = \frac{V^{\frac{2k-n-2|v|}{2}}}{K_n(2k)} \tag{7}$$

for

$$V = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma(\frac{2+2k-n-2|v|}{2}) \Gamma(\frac{1-2k}{2}) \Gamma(2k)}{\Gamma(\frac{2+2k-p-2|v|}{2}) \Gamma(\frac{p-2k}{2})}$$

The proof of this Lemma is given in [10].

Lemma 2.3 *The functions $S_{2k}(x)$ and $R_{2k}(x)$ are homogeneous distributions of order $(2k - n - 2|v|)$ for $Re(2k) < n + 2|v|$. In particular, the B-convolution $S_{2k}(x) * R_{2k}(x)$ exists and is a tempered distribution.*

The proof of this Lemma is given in [10].

Lemma 2.4 The B-convolutions of tempered distributions

- (a) *Let $S_{2k}(x)$ and $S_{2m}(x)$ be defined by (6), then $S_{2k}(x) * S_{2m}(x) = S_{2k+2m}(x)$, where k and m are nonnegative integer,*
- (b) *Let $R_{2k}(x)$ and $R_{2m}(x)$ be defined by (7), then $R_{2k}(x) * R_{2m}(x) = R_{2k+2m}(x)$, where k and m are nonnegative integer,*
- (c) *Let $S_{2k}(x)$ and $S_{2m}(x)$ be defined by (6) and if $S_{2k}(x) * S_{2m}(x) = \delta(x)$, then $S_{2k}(x)$ is an inverse of $S_{2m}(x)$ in the B-convolution algebra, denoted by $S_{2k}(x) = S_{2m}^{*-1}(x)$, moreover $S_{2m}^{*-1}(x)$ is unique.*

The proof of this Lemma is given in [8].

Lemma 2.5 *Given the equation $\diamond_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where \diamond_B^k is the Bessel diamond operator iterated k -times defined by (1). Then $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is an elementary solution of the operator \diamond_B^k .*

The proof of this Lemma is given in [10].

Lemma 2.6 (The elementary solution of The Bessel Helmholtz operator)

Given the equation $(\Delta_B + m^2)^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $(\Delta_B + m^2)^k$ is called the Bessel Helmholtz operator iterated k times, defined by

$$(\Delta_B^k + m^2)^k = (B_{x_1} + B_{x_2} + \dots + B_{x_n} + m^2)^k, \tag{8}$$

n is the dimension of the space \mathbb{R}_n^+ , $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$ for $i = 1, 2, \dots, n$. Then $u(x) = T_{2k}(x, m)$ is an elementary solution of the operator $(\Delta_B + m^2)^k$ where

$$T_{2k}(x, m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} S_{2k+2r}(x) \tag{9}$$

for $S_{2k+2r}(x)$ is defined by (6).

The proof of this Lemma is given in [6].

Lemma 2.7 Given the equation

$$\Delta_B^k u(x) = 0 \text{ for } x \in \mathbb{R}_n^+, \tag{10}$$

where Δ_B^k is the Laplace-Bessel operator iterated k -times defined by (2). Then $u(x) = (-1)^k (S_{2(k-1)}(x))^{(l)}$ is a solution of (10), where l is a nonnegative integer with $l = \frac{n+2|v|-4}{2}$, $n + 2|v| \geq 4$, n is even and $(S_{2(k-1)}(x))^{(l)}$ is a function defined by (6) with l derivatives.

The proof of this Lemma is given in [7].

Lemma 2.8 Given the equation

$$\Delta_B^k u(x) = f(x, u(x)) \text{ for } x \in \mathbb{R}_n^+, \tag{11}$$

where Δ_B^k is the Laplace-Bessel operator iterated k -times defined by (2), f is defined and has continuous first derivative for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of \mathbb{R}_n^+ and $\partial\Omega$ is the boundary of Ω , assume that f is bounded, that is $|f(x, u(x))| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ is an unique solution of (11).

The proof of this Lemma is given in [7].

3 Main Results

These are the main results of the paper.

Theorem 3.1 *Given the nonlinear equation*

$$\diamond_B^k(\Delta_B + m^2)^k u(x) = f(x, \Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x)) \tag{12}$$

where \diamond_B^k is the Bessel diamond operator iterated k times defined by (1), Δ_B^{k-1} is the Laplace-Bessel operator iterated $k-1$ times defined by (2), $(\Delta_B + m^2)^k$ is the Bessel Helmholtz operator iterated k times, \square_B^k is the Bessel ultra-hyperbolic operator iterated k times defined by (3) and m is a positive real number. Let f be defined and having continuous first derivative for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of \mathbb{R}_n^+ and $\partial\Omega$ denotes the boundary of Ω and n is even with $n + 2|v| \geq 4$. Assume that f is bounded, that is for all $x \in \Omega$

$$|f(x, \Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x))| \leq N \tag{13}$$

where N is a constant and the boundary condition for $x \in \partial\Omega$

$$\Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x) = 0. \tag{14}$$

Then we obtain

$$u(x) = (-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * T_{2k}(x, m) * U(x) \tag{15}$$

as a solution of (12), with the boundary condition

$$u(x) = (-1)^{k-1} (S_{2(k-2)}(x))^{(l)} * R_{2k}(x) * T_{2k}(x, m), \tag{16}$$

where $U(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $l = \frac{n+2|v|-4}{2}$, $k = 2, 3, \dots$, $S_{2(k-2)}(x)$, $R_{2k}(x)$ and $T_{2k}(x, m)$ are given by (6), (7) and (9) respectively. Moreover $V(x) = (-1)^{1-k} S_{2(1-k)}(x) * u(x)$ is a solution of the equation

$$\square_B^k(\Delta_B + m^2)^k V(x) = U(x). \tag{17}$$

Furthermore, if we put $k = 1$, $p = n$ and $q = 0$, then $V(x)$ is a solution of the nonhomogeneous Bessel biharmonic equation.

Proof. We have

$$\begin{aligned} \diamond_B^k(\Delta_B + m^2)^k u(x) &= \Delta_B (\Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x)) \\ &= f(x, \Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x)). \end{aligned} \tag{18}$$

Since $u(x)$ has continuous derivatives up to order $6k$ for $k = 1, 2, \dots$, we may assume that for all $x \in \Omega$

$$\Delta_B^{k-1} \square_B^k(\Delta_B + m^2)^k u(x) = U(x). \tag{19}$$

Thus (18) can be written in the form

$$\diamond_B^k(\Delta_B + m^2)^k u(x) = \Delta_B U(x) = f(x, U(x)), \tag{20}$$

by (13), we have for $x \in \Omega$

$$|f(x, U(x))| \leq N \tag{21}$$

and by (14), we obtain for $x \in \Omega$, $U(x) = 0$ or

$$\Delta_B^{k-1} \square_B^k (\Delta_B + m^2)^k u(x) = 0. \tag{22}$$

Then by Lemma 2.8, there exists a unique solution $U(x)$ of (20) which satisfies (21). Now consider the equation (19), we have

$$\Delta_B^{k-1} (-1)^{k-1} S_{2(k-1)}(x) = \delta, \quad \square_B^k R_{2k}(x) = \delta \quad \text{and} \quad (\Delta_B + m^2)^k T_{2k}(x, m) = \delta$$

where δ is the Dirac delta distribution, that is the functions $(-1)^{k-1} S_{2(k-1)}(x)$, $R_{2k}(x)$ and $T_{2k}(x, m)$ are the elementary solution of the operators Δ_B^{k-1} , \square_B^k and $(\Delta_B + m^2)^k$ respectively. The functions $(-1)^{k-1} S_{2(k-1)}(x)$, $R_{2k}(x)$ and $T_{2k}(x, m)$ are defined by (6), (7) and (9) respectively. Hence B-convolving both sides of (19) by $(-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * T_{2k}(x, m)$, we obtain

$$\begin{aligned} U(x) * [(-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * T_{2k}(x, m)] \\ &= \Delta_B^{k-1} \square_B^k (\Delta_B + m^2)^k u(x) * [(-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * T_{2k}(x, m)] \\ &= [\Delta_B^{k-1} (-1)^{k-1} S_{2(k-1)}(x)] * [\square_B^k R_{2k}(x)] * [(\Delta_B + m^2)^k T_{2k}(x, m)] * u(x) \\ &= \delta * \delta * \delta * u(x). \end{aligned}$$

Then

$$u(x) = (-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * T_{2k}(x, m) * U(x) \tag{23}$$

as required. We consider for $x \in \partial\Omega$

$$\Delta_B^{k-1} \square_B^k (\Delta_B + m^2)^k u(x) = 0.$$

By Lemma 2.7, we obtain

$$\square_B^k (\Delta_B + m^2)^k u(x) = (-1)^{k-1} (S_{2(k-2)}(x))^{(l)}.$$

B-convolving both sides of the above equation by $R_{2k}(x) * T_{2k}(x, m)$, we have

$$\begin{aligned} [R_{2k}(x) * T_{2k}(x, m)] * (-1)^{k-1} (S_{2(k-2)}(x))^{(l)} \\ &= [R_{2k}(x) * T_{2k}(x, m)] * \square_B^k (\Delta_B + m^2)^k u(x) \\ &= [\square_B^k R_{2k}(x)] * [(\Delta_B + m^2)^k T_{2k}(x, m)] * u(x) \\ &= \delta * \delta * u(x). \end{aligned}$$

Then for $x \in \partial\Omega$ and $k = 2, 3, \dots$

$$u(x) = R_{2k}(x) * T_{2k}(x, m) * (-1)^{k-1} (S_{2(k-2)}(x))^{(l)} \quad (24)$$

as required. Now B-convolving both sides of (23) by $(-1)^{1-k} S_{2(1-k)}(x)$, we obtain

$$\begin{aligned} & (-1)^{1-k} S_{2(1-k)}(x) * u(x) \\ &= (-1)^{1-k} S_{2(1-k)}(x) * (-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * T_{2k}(x, m) * U(x) \\ &= S_0(x) * R_{2k}(x) * T_{2k}(x, m) * U(x) \\ &= \delta(x) * R_{2k}(x) * T_{2k}(x, m) * U(x) \\ &= R_{2k}(x) * T_{2k}(x, m) * U(x). \end{aligned}$$

Let $V(x) = (-1)^{1-k} S_{2(1-k)}(x) * u(x)$ be given, then $u(x) = (-1)^{k-1} S_{2(k-1)}(x) * V(x)$, and by (19) we have

$$\begin{aligned} U(x) &= \Delta_B^{k-1} \square_B^k (\Delta_B + m^2)^k (-1)^{k-1} S_{2(k-1)}(x) * V(x) \\ &= \Delta_B^{k-1} (-1)^{k-1} S_{2(k-1)}(x) * \square_B^k (\Delta_B + m^2)^k V(x) \\ &= \delta(x) * \square_B^k (\Delta_B + m^2)^k V(x) \\ &= \square_B^k (\Delta_B + m^2)^k V(x). \end{aligned} \quad (25)$$

If we put $k = 1$, $p = n$ and $q = 0$, then the operators \square_B^k reduced to Δ_B . Then $V(x)$ is the solution of the nonhomogeneous Bessel biharmonic equation

$$\Delta_B^2 V(x) = g(x, \Delta_B V(x))$$

where $g(x, \Delta_B V(x)) = U(x) - m^2 \Delta_B V(x)$. □

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