Applied Mathematical Sciences, Vol. 6, 2012, no. 54, 2687 – 2693

Some Properities of Some Subclasses of Univalent Functions

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Abstract. For analytic functions $f(z)$ normalized with $f(0)=0$ and $f^{/}(0)=1$ in the open unit disk U ,a new class $L_1^*(\beta_1,\beta_2,\gamma)$ of $f(z)$ satisfying some conditions with some complex number β_1,β_2 and some real number λ is introduced.The object of the present paper is to discuss some properties for $L_1^*(\beta_1,\beta_2,\gamma)$ of $f(z)$ associated with close-to-convex in U .

Mathematics Subject Classification: 30C45

Keywords: Analytic and univalent functions; Bi-univalent functions; λ -convex functions; Coefficients bounds

1. Introduction and definitions

Let A denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n (a_n \in C) \tag{1.1}$$

which are analytic and univalent in the open unit disc $U := \{z \in C : |z| < 1\}$. Let $R(\alpha)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} f'(z) > \alpha \quad (z \in U)$$

for some real $\alpha (0 \le \alpha < 1)$.A function $f(z) \in R(\alpha)$ is said to be close-to-convex of order α in U (cf.Goodman[1]) .We know that $R(\alpha_2) \subset R(\alpha_1)$ for $0 \le \alpha_1 \le \alpha_2 < 1$ and $R(\alpha) \subset A$ by Noshiro-Warshawski theorem(cf.Duren[2]).

Let $L_1^*(\beta_1, \beta_2, \lambda)$ denote the subclass of A defined as follows:

$$L_1^*(\beta_1, \beta_2, \lambda) = \left\{ f \in A : \left| \frac{f'(z) - 1}{\beta_1 f'(z) + \beta_2} \right| \le \lambda \right\}$$

for some complex β_1, β_2 and for some real λ.Let T denote the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0) \tag{1.2}$$

Further, let $L^*(\beta_1, \beta_2, \lambda)$ denote the subclass of $L_1^*(\beta_1, \beta_2, \lambda)$ by

$$L^*(\beta_1, \beta_2, \lambda) = L_1^*(\beta_1, \beta_2, \lambda) \cap T$$

for some real number $\beta_1 (0 \le \beta_1 \le 1)$ and $\beta_2 (0 < \beta_2 \le 1)$,and for some real number $\lambda (0 < \lambda \le 1)$ .The class $L^*(\beta_1, \beta_2, \lambda)$ was studied by Kim and Lee[3](see also [4,5,6,7]).

We note that:

(i) $L^*(\beta_1, 1, \frac{1-\beta_1}{1+\beta_1^2}) = P^*(\beta_1)$,where $P^*(\beta_1)$ is the class of functions $f(z) \in T$ which satisfy $\operatorname{Re} f'(z) \ge \beta_1$.

The class $P^*(\beta_1)$ was studied by Kim and Lee[3],Sarangi and Uralegaddi[8] and

Al.Amiri[9].

(ii) $L^*(0,1,\lambda)=G^*(\lambda)$,wheres $G^*(\lambda)$ is the class of functions $f(z)\in T$ which satisfy $\left|f^{/}(z)-1\right|\le\lambda$.

The class $G^*(\lambda)$ was introduced by Kim and Lee[3].

(iii) $L^*(1,1,\lambda)=D^*(\lambda)$,where $D^*(\lambda)$ is the class of functions $f(z)\in T$ which satisfy $\left|\dfrac{f^{/}(z)-1}{f^{/}(z)+1}\right|\le\lambda$.

The class $D^*(\lambda)$ was introduced by Kim and Lee[3].

2.Properties of the class $L_1^*(\beta_1,\beta_2,\lambda)$

First result for the class is contained in

Theorem 2.1. A function $f(z)$ defined by (1.3) is in the class $L_1^*(\beta_1,\beta_2,\lambda)$ if and only if

$$\sum_{n=2}^{\infty} n(1+\lambda\,|\,\beta_1\,|)\,|\,a_n\,|\le\lambda\left|\beta_1+\beta_2\right|$$

The result is sharp.

Proof. It follows that

$$\left|\frac{f^{/}(z)-1}{\beta_1 f^{/}(z)+\beta_2}\right|=\left|\frac{\sum_{n=2}^{\infty} na_n z^{n-1}}{(\beta_1+\beta_2)+\sum_{n=2}^{\infty} n\beta_1 a_n z^{n-1}}\right|$$

$$\le\frac{\sum_{n=2}^{\infty} n\left|a_n\right|\left|z^{n-1}\right|}{\left|\beta_1+\beta_2\right|-\sum_{n=2}^{\infty} n\left|\beta_1\right|\left|a_n\right|\left|z^{n-1}\right|}\le\frac{\sum_{n=2}^{\infty} n\left|a_n\right|}{\left|\beta_1+\beta_2\right|-\sum_{n=2}^{\infty} n\left|\beta_1\right|\left|a_n\right|}$$

Therefore,if $f(z)$ satisfies the inequality (2.1),then $f(z)\in L_1^*(\beta_1,\beta_2,\lambda)$.

Conversely,it is simple to verify that if $f(z) \in L_1^*(\beta_1, \beta_2, \lambda)$,then

$$\sum_{n=2}^{\infty} n(1+\lambda|\beta_1|)|a_n| \le \lambda|\beta_1+\beta_2|.$$

Corollary 2.1. If $f(z) \in L_1^*(\beta_1, \beta_2, \lambda)$,then we have $|a_n| \le \dfrac{\lambda|\beta_1+\beta_2|}{n(1+\lambda|\beta_1|)}$ $(n = 2,3,\cdots)$.

Corollary 2.2.[3] A function $f(z)$ defined by (1.2) is in the class $L(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} n(1+\lambda\beta_1)a_n \le \lambda(\beta_1+\beta_2)$$

3.Radius problem for the class $R(\alpha)$

In this section,we discuss some radius problems for the class $R(\alpha)$.To discuss our problems,we need the following lemma for the class $R(\alpha)$.

Lemma 3.1.[10] If $f(z) \in R(\alpha)$,then

$$\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha.$$

Corollary 3.1. If $f(z) \in R(\alpha)$,then $|a_n| \le \dfrac{1-\alpha}{n} \le 1.$

Remark 3.1. By lemma 3.1,we see that if $f(z) \in R(\alpha)$,then

$$\sum_{n=2}^{\infty} n|a_n|^2 \le \sum_{n=2}^{\infty} n|a_n| \le 1-\alpha$$

Remark 3.2. If $f(z) \in R(\alpha)$,then $|a_n| \le \sqrt{\dfrac{1-\alpha}{n}}$.

Now,we derive

Theorem 3.1. If $f(z) \in R(\alpha)$ and $\delta \in C (0 < |\delta| < 1)$,then the function $\dfrac{1}{\delta} f(\delta z)$

belongs to the class $L_1^*(\beta_1,\beta_2,\lambda)$ for $0<|\delta|\leq|\delta_0(\lambda)|$,where $|\delta_0(\lambda)|$ is the smallest positive root of the equation

$$\frac{|\delta|(1+\lambda|\beta_1|)}{1-|\delta|^2}\sqrt{(2-|\delta|^2)(1-\alpha)}=\lambda|\beta_1+\beta_2|$$

in $0<|\delta|<1$.

Proof. For $f(z)\in R(\alpha)$,we see that

$$\frac{1}{\delta}f(\delta z)=z+\sum_{n=2}^{\infty}\delta^{n-1}z^n$$

and

$$\sum_{n=2}^{\infty}n|a_n|^2\leq 1-\alpha$$

To show that $f(z)\in L_1^*(\beta_1,\beta_2,\lambda)$,we need to prove that

$$\sum_{n=2}^{\infty}n(1+\lambda|\beta_1|)|\delta|^{n-1}|a_n|\leq\lambda|\beta_1+\beta_2|$$

from theorem 2.1.Applying Cauchy-Schwarz inequality,we note that

$$\sum_{n=2}^{\infty}n(1+\lambda|\beta_1|)|\delta|^{n-1}|a_n|\leq\frac{1+\lambda|\beta_1|}{|\delta|}\left(\sum_{n=2}^{\infty}n|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}n|a_n|^2\right)^{\frac{1}{2}}$$

$$\leq\frac{1+\lambda|\beta_1|}{|\delta|}\left(\sum_{n=2}^{\infty}n|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha}$$

We note that

$$\sum_{n=2}^{\infty}n|\delta|^{2n}=\frac{|\delta|^4(2-|\delta|^2)}{(1-|\delta|^2)^2}$$

Therefore,we show that

$$\sum_{n=2}^{\infty}n(1+\lambda|\beta_1|)|\delta|^{n-1}|a_n|\leq\frac{|\delta|(1+\lambda|\beta_1|)}{1-|\delta|^2}\sqrt{(2-|\delta|^2)(1-\alpha)}$$

Now,let us consider the complex $\delta(0<|\delta|<1)$ such that

$$\frac{|\delta|(1+\lambda|\beta_1|)}{1-|\delta|^2}\sqrt{(2-|\delta|^2)(1-\alpha)} = \lambda\left|\beta_1+\beta_2\right|.$$

If we define the function $h(|\delta|)$ by

$$h(|\delta|) = |\delta|(1+\lambda|\beta_1|)\sqrt{(2-|\delta|^2)(1-\alpha)} - \lambda\left|\beta_1+\beta_2\right|(1-|\delta|^2),$$

then we have that $h(0) = -\lambda\left|\beta_1+\beta_2\right| < 0$ and $h(1) = (1+\lambda\left|\beta_1\right|)\sqrt{1-\alpha} > 0$ .This means that there exists some δ_0 such that $h(|\delta_0|) = 0 (0 < |\delta_0| < 1)$ .This completes the proof of the theorem.

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Received: December, 2011