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Occasionally Weakly Compatible Mapping in Cone Metric Space

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Abstract. In this paper, we obtain some fixed point theorems for occasionally weakly compatible mappings in cone metric space. Our results extends the result of A.Bhatt, et al.[A. Bhatt, et al., Common fixed point theorems for occasionally weakly compatible mappings under relaxed conditions, Nonlinear Anal., 73(2010) 176-182].

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1. INTRODUCTION

Study of fixed point theorems for a map satisfying a contractive condition that did not require continuity at each point was initiated by Kannan [14] in 1968. Subsequently, several authors established fixed point theorems for a pair of maps. The notion of weakly commuting maps was initiated by Sessa [17]. Jungck [9] gave the concept of compatible maps and showed that weakly commuting mappings are compatible, but converse is not true. During last few years several fixed point theorems have been obtained by various authors utilizing this notion. Jungck further weakened the notion of compatibility by introducing weak compatibility. Pant [15] initiated the study of non-compatible maps and introduced pointwise R-weak commutativity of mappings. Aamri and Moutawakil [1] introduced the (E.A) property and thus generalized the concept of non-compatible maps. The results obtained in the metric fixed point theory by using the notion of non-compatible maps or the property (E.A) are very interesting. Lastly, Al-Thagafi and Shahzad [4] defined the concept of occasionally weakly compatible mappings which is more general than the concept weakly compatible maps. Recently Chandra and Bhatt [16, 7] have proved the results of Jungck and Rhoades [11, 12] in probabilistic semi-metric space and for pairs of multivalued mappings. Bhatt, Chandra and Sahu have given application of these results in dynamical system in [2]. Guang and Xian [8] generalized the concept of a metric space, repacing the real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions.

In this paper we prove a common fixed point theorem of occasionally weakly compatible mappings in cone metric spaces. Our results extends the result of A.Bhatt, H. Chandra and D.R.Sahu [2].

Consistent with Guang and Xian [8], the following definitions and motivations are arranged sequentially.

Definition 1.1. Let E be a real Banach space. A subset P of E is called a *cone* if and only if.

(a) P is closed, nonempty and $P \neq \{0\}$.
(b) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;
(c) $p \cap (-p) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$.

$$0 \leq x \leq y \Rightarrow ||x|| \leq K||y||. \tag{1}$$

The least positive number satisfying the above inequality is called the *normal constant of* P, while $x << y$ stands for $y - x \in intP$(interior of P).

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric in X and (X, d) is called a *cone metric space.*The concept of a cone metric space is more general than that of a metric space.

Definition 1.3. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(a) A *Cauchy sequence* if for every c in E with $c >> 0$, there is N such that for all $n, m > N$, $d(x_n, x_m) << c$.
(b) A *Convergent sequence* if for every c in E with $c >> 0$, there is N such that for all $n > N, d(x_n, x) << c$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence is unique provided P is a normal cone with normal constant K.

Definition 1.4. ([11]) Let X be a set and let f, g be two self-mappings of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

The following concept [4] is a proper generalization of nontrivial weakly compatible maps which do have a coincidence point.

Definition 1.5. Two self-maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

We state the following lemma which is proved in [11].

Lemma 1.1. Let X be a set, f, g owc self-mappings of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

2. Section II

In this section, we prove some fixed point theorems for a pair of two occasionally weakly commuting mappings on cone metric space.
Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying the condition $\phi(t) < t$ for each $t > 0$. We now prove the following theorem.

Theorem 2.1. Let (X, d) be a cone metric space and P be a normal cone. Suppose f and g are occasionally weakly compatible (owc) self-mappings of X and satisfying the following conditions:

$$d(fx, fy) \leq \phi\left(\max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}\right), \tag{2}$$

$\forall x, y \in X$. Then f and g have a unique common fixed point.

Proof. Since f and g are owc, there exists a point u in X such that $fu = gu$, $fgu = gfu$. We claim fu is the unique common fixed point of f and g. We first assert that fu is a fixed point of f. For, if $ffu \neq fu$ then from (2), we get

$$\begin{aligned} d(fu, ffu) &\leq \phi\left(\max\{d(gu, gfu), d(gu, ffu), d(gfu, fu), d(gfu, ffu)\}\right) \\ &= \phi\left(\max\{d(fu, ffu), d(ffu, fu)\}\right) < d(ffu, fu). \end{aligned}$$

This is a contradiction. Hence $ffu = fu$ and $ffu = fgu = gfu = fu$. Thus, fu is a common fixed point of f and g. For uniqueness, suppose that $u, v \in X$ such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then (2) gives

$$\begin{aligned} d(u, v) &= d(fu, fv) \\ &\leq \phi\left(\max\{d(gu, gv), d(gu, fv), d(gv, fu), d(gv, fv)\}\right) \\ &= \phi\left(\max\{d(u, v), d(v, u)\}\right) < d(u, v). \end{aligned}$$

This is a contradiction. Therefore, $u = v$. Therefore, the common fixed point of f and g is unique. □

Example 2.1. *Let* $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0 \subset R^2,\}$ *and define* $d : R \times R \to E$ *by* $d(x, y) = (|x - y|, \alpha|x - y|)$,

where $\alpha > 0$ is a constant. Define $f, g : X \to X$ by

$$f(x) = \frac{1+2x}{3}, \text{ and } g(x) = \frac{1+4x}{5}, x \in X$$

Clearly, (X, d) is a cone metric space. f and g are owc and f, g satisfy the condition (2). Also 1 is the unique common fixed point of f and g.

Corollary 2.1. Let (X, d) be a cone metric space and P be a normal cone. Suppose f and g are occasionally weakly compatible (owc) self-mappings of X and satisfying the following condition:

(3) $$d(fx, fy) \leq \phi\{d(gx, gy)\}, \forall x, y \in X,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function satisfying the condition $\phi(t) < t$ for each $t > 0$. Then f and g have a unique common fixed point.

The proof of following theorem can be easily obtained by replacing condition (2) by condition (4) in the proof of Theorem 2.1.

Theorem 2.2. Let (X, d) be a cone metric space and P be a normal cone. Suppose f and g are occasionally weakly compatible (owc) self-mappings of X and satisfying the following condition:

(4) $$d(fx, fy) < max\,\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}$$

for all $x, y \in X$, $x \neq y$. Then f and g have a unique common fixed point.

Theorem 2.3. Let (X, d) be a cone metric space and P be a normal cone. Suppose f and g are occasionally weakly compatible (owc) self-mappings of X and satisfying the following conditions:

(5) $$f(X) \subset g(X),$$

(6) $$\begin{aligned} d(fx, fy) \leq ad(gx, gy) &+ b \max\{d(fx, gx), d(fy, gy)\} \\ &+ c \max\{d(gx, gy), d(gx, fx), d(gy, fy)\}, \end{aligned}$$

for all $x, y \in X$, where $a, b, c > 0, a + b + c = 1$. Then f and g have a unique common fixed point.

Proof. It follows from the given assumption that there exists a point $x \in X$ such that $fx = gx$. Suppose there exists another point $y \in X$ for which $fy = gy$. Then, from (7), we have

$$\begin{aligned} d(fx, fy) &\leq ad(fx, fy) + b \max\{0, 0\} + c \max\{d(fx, fy), 0, 0\}, \\ &= (a + c)d(fx, fy). \end{aligned}$$

Since $a+c<1$, the above inequality implies that $d(fx,fy)=0$, which is turn implies that $fx=fy$. Therefore fx is unique. From Lemma 1.1, f and g have a unique fixed point. □

3. Section III

In this section, we prove some fixed point theorems for a pair of four occasionally weakly commuting mappings on cone metric space.

Theorem 3.1. Let (X,d) be a cone metric space and P be a normal cone. Suppose that f,g,S,T are self-mappings of X and that the pairs $\{f,S\}$ and $\{g,T\}$ are each owc. If

$$d(fx,gy) < \max\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), d(Sx,gy), d(Ty,fx)\}, \tag{7}$$

for each $x,y\in X$ for which $fx\neq gy$, then f,g,S and T have a unique common fixed point.

Proof. Since the pairs $\{f,S\}$ and $\{g,T\}$ are each owc, there exist points $x,y\in X$ such that $fx=Sx$ and $gy=Ty$. We claim that $fx=gy$. For, otherwise, by (7),

$$d(fx,gy) < \max\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), d(Sx,gy), d(Ty,fx)\}$$

Since $fx=Sx=w$ and $gy=Ty=z$ are points of coincidece of $\{f,S\}$ and $\{g,T\}$, respectively, hence,

$$d(fx,gy) < max\,\{d(fx,gy), d(gy,fx)\} = d(fx,gy).$$

This is a contradiction. Therefore, $fx=gy$; i.e., $fx=Sx=gy=Ty$. Moreover, if there is another point z such that $fz=Sz$, then, using (7) it follows that $fz=Sz=gy=Ty$, or $fx=fz$. Hence $w=fx=Sx$ is the unique point of coincidence of f and S. By Lemma 1.1, w is the unique common fixed point of f and S. Similarly, there is a unique point $z\in X$ such that $z=gz=Tz$. Suppose that $w\neq z$. Using (7) we get,

$$d(w,z) = d(fw,gz) < max\,\{d(w,z), d(z,w)\} = d(w,z).$$

This is a contradiction. Therefore, $w=z$ and w is the unique common fixed point of f,g,S and T. □

Let $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be a control function such that ϕ is continuous, nondecreasing, $\phi(2t)\leq 2\phi(t)$ and $\phi(t)=0$ iff $t=0$. Let $\psi:\mathbb{R}^+\to\mathbb{R}^+$ be another function such that $\psi(t)<t$ for each $t>0$.

Theorem 3.2. Let (X,d) be a cone metric space and P be a normal cone. Suppose that f,g,S,T are self-mappings of X and that the pairs $\{f,S\}$ and $\{g,T\}$ are each owc. If

$$\phi(d(fx,gy)) \leq \psi(M_\phi(x,y)), \tag{8}$$

where

$$M_\phi(x,y) := \max\{\phi(d(Sx,Ty)), \phi(d(Sx,fx)), \phi((d(gy,Ty)),$$
$$[\phi(d(fx,Ty)) + \phi(d(Sx,gy))]/2\},$$

for all $x, y \in X$. Then f, g, S, and T have a unique common fixed point.

Proof. By hypothesis there exist points $x, y \in X$ for which $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (8), we get

$$\begin{aligned} 0 < \phi((d(fx,gy)) &\leq \psi(M_\phi(x,y)) \\ &= \psi(\phi(d(fx,gy)) \\ &< \phi((d(fx,gy)), \end{aligned}$$

which is a contradiction. Therefore $\phi((d(fx,gy)) = 0$, hence $d(fx,gy) = 0$, which implies that $fx = gy$. Now from the repeated use of the condition (8) we can show that f, g, S and T have a unique common fixed point. □

Remark 3.1. As an application of Corollary 2.1, the existence and uniqueness of a common solution of the functional equations arising in dynamic programming can be established which extends Theorem 4.1 ([2]).

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