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Common Fixed Points of Weakly Compatible Maps in Symmetric Spaces

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Abstract. In this paper we obtain common fixed point theorems of weakly compatible maps on symmetric spaces. We prove that if S and T are weakly compatible maps satisfying property (E-A) along with strict contractive conditions, then they have common fixed points. Since these results are obtained without using full force of metric, they are improved generalization of results obtained by Pant ([10], [11]).

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1. INTRODUCTION

Presence of fixed points is an intrinsic property of a self-map. However, there are many conditions for the existence of such points which involve a mixture of algebraic, order theoretic and topological properties of mappings or its domain. In 1976 Jungck [5] proved a common fixed point theorem for commuting maps. This theorem generalizes the Banach's fixed point theorem. It has many applications. But the theorem suffers from draw back that it requires commuting maps to be continuous. Thereafter several papers (see [6], [7], [9], [10]) were published involving contractive definition that do not require the continuity. The result of Jungck was further generalized and extended in various ways. Sessa [13] defined weak commutativity maps. Subsequently, Jungck [6] introduced the notion of compatibility, which is more general than that of weak commutativity. In 1994 Pant [9] introduced the concept of R-weakly commuting maps and pointwise R-weakly commuting maps.

In view of a paper by Pant [10], it may be observed that pointwise R - weak commutativity is

(i) equivalent to commutativity at coincident points; and
(ii) a necessary, hence minimal, condition for the existence of common fixed points of contractive type mappings.

In 1998, Jungck and Rhoades [7] introduced the notion of weakly compatible maps. All these authors proved certain fixed point theorems for the defined maps in the setting of metric spaces. In this paper our main objective is to obtain some results on fixed points in symmetric spaces under strict contrative conditions. Our results are improved generalization of the results obtained by Pant ([10], [11]). The relevant definitions and motivations are as follows.

Definition 1.1. Let S and T be two self-mappings of a metric space (X, d). Then S and T said to be Compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0.$$

whenever x_n is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ exists and belongs to X.

Definition 1.2. Two self- maps S and T of a metric space (X, d) are said to be weakly compatible if they commute at the coincidence points. i.e.

$$\text{if } Tu = Su \text{ for some } u \in X$$
$$\text{then } TSu = STu.$$

It is easy to see that compatible maps are weakly compatible.

Example 1.1. Let $X = [0, 1]$ and d denote the usual metric on X. Define S, $T : X \to X$ as

$$Sx = \begin{cases} x & \text{, when x is an rational number} \\ 0 & \text{, when x is an irrational number} \end{cases}$$

$$Tx = \begin{cases} 0 & \text{, when x is an rational number} \\ 1 & \text{, when x is an irrational number.} \end{cases}$$

In this example, it is easy to see that S and T are compatible as well as weakly compatible. But S and T are both discontinuous. However, weakly compatible maps need not be compatible. A simple example is the following.

Example 1.2. Let$X = [0, 1]$ and d denote the usual metric on X.
Define S, $T : X \to X$ as

$$\text{Sx} = \frac{1+x}{2} \text{ for } x \in [0, 1]$$

$$Tx = \begin{cases} \frac{1-x}{2} & \text{, when } [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ \frac{3}{4} & \text{, when x } = \frac{1}{2}. \end{cases}$$

Then $S(x) = T(x)$. This implis that x=0 and (ST)(0)=(TS)(0)=$\frac{3}{4}$. Hence S and T are weakly compatible. To show that S and T are non-compatible. Consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$. Then $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = \frac{1}{2}$. But $\lim\limits_{n\to\infty} d(STx_n, TSx_n) = \frac{1}{2} \neq 0$. This shows that S and T are non-compatible.

Definition 1.3. Let S and T be two self-mappings of a metric space (X, d). Then S and T are pointwise R-Weakly Commuting if given $x \in X$ $\exists R > 0$ such that

$$d(STx, TSx) \leq Rd(Sx, Tx).$$

Definition 1.4. A pair of self-mappings S and T on a symmetric space (X, d) is said to enjoy property **(E-A)** if there exists a sequence $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is clear that noncompatible pairs satisfy property **(E-A)**. In the papers ([10], [11]) the following results have been proved by Pant.

Theorem 1.1. Let S and T be non- compatible pointwise R-weakly commuting self mappings of a metric space (X, d), satisfying

$$\overline{SX} \subset TX \tag{1}$$

$$d(Sx, Sy) \leq kd(Tx, Ty),\ k \geq 0 \tag{2}$$

$$d(Sx, S^2x) \neq max\{d(Sx, TSx),\ d(S^2x, TSx)\}, \tag{3}$$

whenever the right-hand side is non-zero. Then S and T have a common fixed point.

Theorem 1.2. Let S and T be non- compatible point wise R- weakly commuting self mappings of a metric space (X, d) s.t.

$$SX \subset TX \tag{4}$$

$$d(Sx, Sy) < Max\{d(Tx, Ty), \frac{k}{2}[d(Sx, Tx) + d(Sy, Ty)], \frac{1}{2}[d(Sy, Tx) + d(Sx, Ty)]\} \tag{5}$$

where$1 \leq$ k$<$ 2. If the range of S or T is a complete subspace of X then S and T have a unique fixed point.

In this paper we prove certain results relating to common fixed point of pair of weakly compatible maps on symmetric spaces. Our results are valid in more general setting of symmetric spaces. Further, the class of pair of maps satisfying conditions in([10], [11])is contained in that in this paper. Hence the results of this paper are improved generalization of the results given in ([10], [11]).

Definition 1.5. Let X be a non-empty set. A **symmetric** on a set X is a real valued function $d : X \times X \to$ R such that

(i). $d(x, y) \geq 0, \forall x, y \in X$.
(ii). $d(x, y) = 0 \Leftrightarrow x = y$.
(iii). $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X. Let $B(x, \epsilon)$=$\{y \in X : d(x, y) < \epsilon\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U, B(x, \epsilon) \subset U$ for some $\epsilon > 0$.

A symmetric d is a semi-metric if for each $x \in$ X and each$\in> 0$, $B(x, \epsilon)$ is a neighborhood of x in the topology $t(d)$. There are several concepts of completeness in this setting. A sequence is called d-Cauchy if it satisfies the usual metric condition.

Definition 1.6. Let (X, d) be a symmetric (semi-metric) space.

(i). (X, d) is S-Complete if for every d-Cauchy sequence $\{x_n\}$ there exists x in X with lim $d(x_n, x) = 0$.

(ii). (X, d) is d-Cauchy Complete if for every d-Cauchy sequence $\{x_n\}$ there exists x in X with lim $x_n = x$ with respect to $t(d)$.

(iii). $S : X \to X$ is d-Continuous if lim $d(x_n, x) = 0$ implies lim $d(Sx_n, Sx) = 0$.

(iv). $S : X \to X$ is $t(d)$ continuous if lim $x_n = x$ with respect to $t(d)$ implies lim $S(x_n) = Sx$ with respect to $t(d)$.

The following two axioms W_1 and W_2 were given by Wilson [14].

Definition 1.7. Let (X, d) be a symmetric (semi-metric) space.

(W_1). Given $\{x_n\}$, x and y in X, $d(x_n, x)$$\to 0$ and $d(x_n, y)$$\to 0 \Rightarrow$ x=y.

(W_2). Given $\{x_n\}$, $\{y_n\}$ and an x in X, $d(x_n, x)$$\to 0$ and $d(x_n, y_n) \to 0 \Rightarrow$x=y.

2. Main Results

Let (X, d) be a symmetric (semi- metric) space which satisfies W_1 condition. Let S and T be a self mappings of a symmetric space (X, d) and let $\overline{SX}$ denote the closure of the range of S. We now prove the following theorem.

Theorem 2.1. Let S and T are weakly compatible self mappings of a symmetric (semi-metric) space (X, d) and S and T satisfy property **(E-A)** such that

$$\overline{SX} \subset TX \tag{6}$$

$$d(Sx, Sy) \leq kd(Tx, Ty),\ k \geq 0 \tag{7}$$

$$d(x, Tx) < Max\left\{d(x, Sx), d(Tx, Sx)\right\}, \tag{8}$$

whenever the right-hand side is non-zero. Then S and T have a common fixed point. Moreover all fixed points of S are fixed points of T.

Proof. S and T satisfy property **(E-A)**, there exists a sequence $\{x_n\}$ such that $Sx_n \to t$ and $Tx_n \to t$ for some t in X. Then since $t \in \overline{SX}$ and $\overline{SX} \subset TX$ there exists x_0 in X such that $t = Tx_0$. By (7) we get

$$d(Sx_n, Sx_0) \leq kd(Tx_n, T_0).$$

On letting $n \to \infty$

$$\lim_{n\to\infty} d(Sx_n, Sx_0) = 0.$$

Now Since $Sx_n \to t = Tx_0$ and $Sx_n \to Sx_0$. Hence by condition W_1 we get $Sx_0 = Tx_0$. It follows that, weak compatibility of S and T implies that $STx_0 = TSx_0$. Also $SSx_0 = STx_0 = TSx_0 = TTx_0$.
We claim that $TSx_0 = Sx_0$. If not by virtue (8) we get

$$\begin{aligned} d(Sx_0, TSx_0) &< Max\left\{d(Sx_0, SSx_0), d(TSx_0, SSx_0)\right\}, \\ &= d(Sx_0, SSx_0) = d(Sx_0, TSx_0). \end{aligned}$$

This is a contradiction. Hence $Sx_0 = SSx_0 = TSx_0$ and Sx_0 is a common fixed point of S and T.
We claim that every fixed point of S is a fixed point of T. Let $Sy_0 = y_0$. Suppose $Ty_0 \neq y_0$, Then since Max$\{d(y_0, Sy_0)\} = d(y_0, Ty_0) > 0$. Hence by condition (8) we get

$$\begin{aligned} d(y_0, Ty_0) &< Max\left\{d(y_0, Sy_0), d(Ty_0, Sy_0)\right\} \\ &< Max\left\{d(y_0, y_0), d(Ty_0, y_0)\right\} \\ &< d(Ty_0, y_0). \end{aligned}$$

So $Ty_0 = y_0$. This is a contradiction. Hence every fixed point of S is also a fixed point of T.

□

Remark 2.1. The following example shows that the conclusion of the above theorem may fail if we relax the condition (8) of the above theorem.

Example 2.1. Let $X = [0, 1]$ with the symmetric (semi-metric)

$$d(x, y) = (x - y)^2$$

Define $S, T : X \to X$ as

$$Sx = \begin{cases} \frac{1+x}{2} & , \text{ When } x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \\ \frac{1}{2} & , \text{When } x = \frac{1}{2} \end{cases}$$

$$Tx = 1 - x, 0 \leq x \leq 1.$$

Then both S and T satisfy conditions (6) and (7) of the above theorem. But they do not satisfy the condition (8) of theorem 2.1 as $d(1, T1) = 1$ and Max $\{d(1, S1), d(T1, S1)\} = 1$. Also$\frac{1}{2}$ is the only common fixed point of S and T and 1 is a fixed point of S but is not a fixed point of T. Thus the condition (8) of the above theorem minimal condition for validity of it's conclusions.

Theorem 2.2. Let S and T are weakly compatible self mappings of a symmetric (semi-metric) space (X, d) and S and T satisfy property **(E-A)** such that

$$SX \subset TX \tag{9}$$

$$d(Sx, Sy) < max \left\{ d(Tx, Ty), \frac{1}{2} \left[d(Sy, Tx) + d(Sx, Ty) \right] \right\} \tag{10}$$

whenever right hand side is non-zero. If the range of S or T is a S-complete subspace of X. Then S and T have a *unique fixed point*.

Proof. Since S and T satisfy property **(E-A)**, there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X. Suppose that the range of S is a S-complete subspace of X. Since $SX \subset TX$ and t$\in X$. There exists some point $x_0 \in X$ such that $t = Tx_0$ where $t = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n$.
If $Sx_0 \neq Tx_0$. Then by condition (10) we get

$$\begin{aligned} d(Tx_0, Sx_0) &< max \left\{ d(Tx_0, Tx_0) \frac{1}{2} \left[d(Sx_0, Tx_0) + d(Sx_0, Tx_0) \right] \right\} \\ &= \frac{1}{2} [d(Sx_0, Tx_0)] < d(Tx_0, Sx_0). \end{aligned}$$

This is a contradiction. Hence $Sx_0 = Tx_0$. Weak compatibility of S and T implies that if $Sx_0 = Tx_0$ for some $x_0 \in X$.Then $TSx_0 = STx_0$. This gives

$SSx_0 = STx_0 = TSx_0 = TTx_0$. If $Sx_0 \neq SSx_0$ then by (10) we have

$$d(Sx_0, SSx_0) < max\left\{d(Tx_0, TSx_0)\frac{1}{2}\left[d(SSx_0, Tx_0) + d(Sx_0, TSx_0)\right]\right\}$$
$$= d(Sx_0, SSx_0).$$

This is again a contradiction. Hence $Sx_0 = SSx_0$ and $Sx_0 = SSx_0 = STx_0 = TSx_0$. Hence Sx_0 is a common fixed point of S and T. The case when range of X is a complete subspace of X is similar to the above case. When $SX \subset TX$. From condition (10) of the above theorem uniqueness of the common fixed point follows easily. Hence we have the theorem. □

We now give an example to illustrate the above theorem.

Example 2.2. *Let $X = [0, 1]$ with the symmetric(semi-metric)*

$$d(x, y) = (x - y)^2$$

Define $S, T : X \to X$ as

$$Sx = \frac{1 + x}{2}, if\ 0 \leq x \leq 1$$

$$Tx = \begin{cases} \frac{1-x}{2} & , when\ if\ 0 < x < \frac{3}{4}\ and\ \frac{3}{4} < x < 1, \\ 0 & , if\ x = 0 \\ \frac{3}{4} & , when\ x = \frac{3}{4} \\ 1 & , if\ x = 1. \end{cases}$$

Then S and T satisfying all the conditions of the above theorems and have common fixed point $x = 1$.

Remark 2.2. It is pertinent to note that the condition (8) given in the above theorem is a simplification and improvement over the condition (3) given in theorem (1.6) by Pant.

References

[1] M. Aamri and D. El Moutawakil, *Some new Common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl.**270**(2002), 181-188.

[2] M. Aamri and D. El Moutawakil, *Common fixed points for weakly compatible maps in symmetric spaces with application to probabilistic spaces*,Applied Mathematics E-Notes,**5**(2005), 171-175.

[3] H. Chandra and A .Bhatt, *Fixed point theorems for pairs of Multivalued mappings*, Kochi Journal of Mathematcs **04**, 101-108, 2009.

[4] T.L.Hicks and B.E.Rhoades, *Fixed point theory in symmetric spaces with applications to probabilistic spaces*, Nonlinear Analysis. **36**(1999), 331-344.

[5] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83**(1976), 261-263.

[6] G. Jungck, *Compatible mappings and common fixed points(2)*, Int. J. Math. & Math. Sci., **11**(1988), 285-288.

[7] G. Jungck B.E.Rhoades , *Fixed points for set valued functions without continuity*, Indian. J.Pure.Appl.Math. **29(3)**(1988), 227-238.

[8] W.A.Kirk and Brailey Sims , *Handbook of Metric fixed point theory*, Kluwer Academic Publishers,**(**2001).

[9] R. P. Pant, *Common fixed points of non commuting mappings*, J.Math.Anal.Appl. Soc. **188**(1994), 436-440.

[10] R. P. Pant, *Common fixed points of Lipshitz Type Mapping Pairs*, J.Math.Anal.Appl. Soc. **240**(1998), 280-283.

[11] R. P. Pant and V.Pant *Common fixed points under strict contractive conditions*, J.Math.Anal.Appl. Soc. **248**(2000), 327-332.

[12] S.P.Singh and B.E.Meade, *On common fixed point theorems*, Bull. Austral. Math. Soc. **16**(1977), 49-53.

[13] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32**(1982), 149-153.

[14] W.A.Wilson, *On semi metric spaces*, Amer.J.Math. **53**(1931), 361-373.

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