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Qualitative Analysis of SIS Epidemic Models in Two Competing Species

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Abstract

Almost all mathematical models of infectious diseases depend on subdividing the population into a set of distinctive classes dependent upon experience with respect to the relevant disease. In our work we will classify individuals as either a susceptible individual S or an infected individual I. Two SIS epidemic models in two competing species are formulated and analyzed. The two species are both subject to a disease. We analyze two different types of incidence, standard incidence and mass action incidence. Thresholds are identified which determine the existence of equilibria, when the populations will survive and when the disease remains endemic. Also stability results are proved. Using Hopf bifurcation theory some results of complicated dynamic behavior of the models are shown. With the interinfection rate of disease between the two species as a bifurcation parameter, it is shown that the model exhibits a Hopf bifurcation leading to a family of periodic solutions.

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1 Introduction

Initially, models for ecological interactions and models for infectious diseases were developed separately. It has been observed that a strong interaction may arise between these factors. When the infection does not lead to immunity, so that infectives become susceptible again after recovery, the disease is called an SIS disease.

It must be noted that the behavior of the model becomes much simpler [9] if one assumes that infection incidence is proportional to the infected fractions in each species (standard incidence) and not to the densities of infectives (mass action incidence). For human diseases the contact rate seems to be only very weakly dependent on the population size, so that the standard incidence is a better approximation.

Models for two species which share a disease without competition have been discussed in some papers. Epidemic models in competing species have also been studied previously. Anderson and May [1] considered a host-competitor-pathogen model which involves two direct competitors, one subject to a pathogen. They examined the effect of a pathogen on conventional competition. In a thorough study of an SIS competing species model with mass action incidence, density-independent death rates, and disease-related deaths, Bowers and Turner [2] developed criteria to show how the forces of competition and infection combine. Venturino [13] analyzed the dynamics of two competing species when one of them is subject to a disease. In his model with mass action incidence, he obtained limit cycles. Han et al. [9] studied an SIRS epidemic model of two competitive species without disease-related deaths. They analyzed the effect of inter-infection of disease on the dynamic behaviors of the model. Van den Driessche and Zeeman [12] investigated the interaction of disease and competition dynamics in a system of two competing species in which only one species is susceptible to disease. Saenz and Hethcote [10] considered some models of SIS, SIR and SIRS type with frequency-dependent incidence. Tompkins, White, and Boots [11] used a variation of the model of Bowers and Turner [2] with density-independent death rates and mass action incidence to study the effects of a parapox virus in competing squirrel species in the United Kingdom.

In this paper, we consider the following competition model

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(\varepsilon_1 - \sigma_1 N_1 - \alpha_1 N_2) \\ \frac{dN_2}{dt} &= N_2(\varepsilon_2 - \sigma_2 N_2 - \alpha_2 N_1) \end{aligned}$$

We analyze SIS epidemic models in the two competing species with the standard incidence and the mass action incidence were both species can be infected.

The paper is organized as follows: In section 2, we formulate the model and explain some basic results on the competition model. In section 3, we analyze the standard incidence model. In section 4, we analyze the mass action incidence model. Finally, in section 5, a conclusion will be given to summarize our results.

2 Formulation of the Model

We consider two competing species that survive in the same habitat on the same resources [3, P.30]. For example, sheep and cows grazing on the same pasture.

First, we need to define the following notation:

$N_1(t)$ and $N_2(t)$ are the densities of the two species at time t.

ε_i are the intrinsic growth rates.

σ_i are the strength of the *intraspecific competition* (the competition within the species).

α_i are the strength of the *interspecific competition* (the competition between the two species, for instance, α_1 is the amount by which one unit of species 2 decreases the per capita growth rate of species 1) .

$\dfrac{\varepsilon_i}{\sigma_i}$ are the carrying capacity of each species in isolation.

We now assume that both species can be infected by a common pathogen, whose cycle follow an SIS scheme, i.e. following recovery an individual become susceptible and can be infected again. Each species will then be divided in a susceptible part S_1, S_2 and an infected class I_1, I_2.

We let

β_{ii} be the intrainfection rates of disease in species i, and

β_{ij} $(i \neq j)$ be the interinfection rate of disease between the two species, and

γ_i are the recovery rate.

Here we assume that all the parameter are nonnegative. For the total population size N_i, we have

$$I_i + S_i = N_i \ , i = 1, 2$$

2.1 Basic result on competition model

Consider first the competition model :

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(\varepsilon_1 - \sigma_1 N_1 - \alpha_1 N_2) \\ \frac{dN_2}{dt} &= N_2(\varepsilon_2 - \sigma_2 N_2 - \alpha_2 N_1) \end{aligned} \tag{2.1}$$

The following can be found in [5]. There are four equilibrium points

$$E_0 = (0,0) \quad , \quad E_1 = \left(\frac{\varepsilon_1}{\sigma_1}, 0\right) \quad , \quad E_2 = \left(0, \frac{\varepsilon_2}{\sigma_2}\right) \quad , \quad E_3 = (N_{1E}, N_{2E})$$

where

$$N_{1E} = \frac{\sigma_2\varepsilon_1 - \alpha_1\varepsilon_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, N_{2E} = \frac{\sigma_1\varepsilon_2 - \alpha_2\varepsilon_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}$$

Lemma 1 *System (2.1) always has the equilibria E_0, E_1 and E_2. Assuming that all parameters are positive, E_0 is always unstable. As for the existence of an internal equilibrium and the stability of them all, there are 4 generic cases:*

1. If $\frac{\varepsilon_1}{\sigma_1} < \frac{\varepsilon_2}{\alpha_2}$, $\frac{\varepsilon_2}{\sigma_2} < \frac{\varepsilon_1}{\alpha_1}$ and $\frac{\sigma_1}{\alpha_1} > \frac{\alpha_2}{\sigma_2}$, there exists also a unique internal equilibrium E_3. E_3 is globally asymptotically stable. In this case E_1 and E_2 are unstable.

2. If $\frac{\varepsilon_1}{\sigma_1} > \frac{\varepsilon_2}{\alpha_2}$, $\frac{\varepsilon_2}{\sigma_2} > \frac{\varepsilon_1}{\alpha_1}$ and $\frac{\sigma_1}{\alpha_1} < \frac{\alpha_2}{\sigma_2}$, there exists a unique internal equilibrium E_3, which is a saddle point. Both E_1 and E_2 are locally asymptotically stable.

3. If $\frac{\varepsilon_1}{\sigma_1} > \frac{\varepsilon_2}{\alpha_2}$ and $\frac{\varepsilon_2}{\sigma_2} < \frac{\varepsilon_1}{\alpha_1}$, there is no internal equilibrium, $E_1$is globally asymptotically stable and E_2 is unstable.

4. If $\frac{\varepsilon_1}{\sigma_1} < \frac{\varepsilon_2}{\alpha_2}$ and $\frac{\varepsilon_2}{\sigma_2} > \frac{\varepsilon_1}{\alpha_1}$, there is no internal equilibrium, $E_2$is globally asymptotically stable and E_1 is unstable.

3 Competition SIS Model with Standard Incidence

Consider the following autonomous competition SIS model with standard incidence:

$$\begin{aligned}
\frac{dI_1}{dt} &= \left(\frac{\beta_{11}I_1}{N_1} + \frac{\beta_{12}I_2}{N_2}\right) S_1 - (\gamma_1 + \sigma_1 N_1 + \alpha_1 N_2) I_1 \\
\frac{dS_1}{dt} &= \varepsilon_1 N_1 + \gamma_1 I_1 - \left(\sigma_1 N_1 + \alpha_1 N_2 + \frac{\beta_{11}I_1}{N_1} + \frac{\beta_{12}I_2}{N_2}\right) S_1 \\
\frac{dN_1}{dt} &= N_1(\varepsilon_1 - \sigma_1 N_1 - \alpha_1 N_2) \\
\frac{dI_2}{dt} &= \left(\frac{\beta_{22}I_2}{N_2} + \frac{\beta_{21}I_1}{N_1}\right) S_2 - (\gamma_2 + \sigma_2 N_2 + \alpha_2 N_1) I_2 \\
\frac{dS_2}{dt} &= \varepsilon_2 N_2 + \gamma_2 I_2 - \left(\sigma_2 N_2 + \alpha_2 N_1 + \frac{\beta_{22}I_2}{N_2} + \frac{\beta_{21}I_1}{N_1}\right) S_2 \\
\frac{dN_2}{dt} &= N_2(\varepsilon_2 - \sigma_2 N_2 - \alpha_2 N_1)
\end{aligned} \tag{3.1}$$

Let $X_i = \dfrac{I_i}{N_i}$ and $Y_i = \dfrac{S_i}{N_i}$ denote the fractions of the classes I_i and S_i in the population, respectively $i = 1, 2$.System(3.1) is converted to

$$\begin{aligned}
\dot{X}_1 &= \beta_{12} Y_1 X_2 - (\varepsilon_1 + \gamma_1 - \beta_{11} Y_1) X_1 \\
\dot{Y}_1 &= \varepsilon_1 + \gamma_1 X_1 - (\varepsilon_1 + \beta_{11} X_1 + \beta_{12} X_2) Y_1 \\
\dot{N}_1 &= N_1(\varepsilon_1 - \sigma_1 N_1 - \alpha_1 N_2) \\
\dot{X}_2 &= \beta_{21} Y_2 X_1 - (\varepsilon_2 + \gamma_2 - \beta_{22} Y_2) X_2 \\
\dot{Y}_2 &= \varepsilon_2 + \gamma_2 X_2 - (\varepsilon_2 + \beta_{22} X_2 + \beta_{21} X_1) Y_2 \\
\dot{N}_2 &= N_2(\varepsilon_2 - \sigma_2 N_2 - \alpha_2 N_1)
\end{aligned} \tag{3.2}$$

System (3.2) comprises six equations, but only four are necessary, since $X_i + Y_i = 1$. We choose to use as variables N_i, X_i and letting $Y_i = 1 - X_i$ for $i = 1, 2$, obtaining the following 4-dimensional system:

$$\begin{aligned}
\dot{X}_1 &= \beta_{12} X_2 - (\gamma_1 + \varepsilon_1 - \beta_{11} + \beta_{11} X_1 + \beta_{12} X_2) X_1 \\
\dot{N}_1 &= N_1(\varepsilon_1 - \sigma_1 N_1 - \alpha_1 N_2) \\
\dot{X}_2 &= \beta_{21} X_1 - (\gamma_2 + \varepsilon_2 - \beta_{22} + \beta_{22} X_2 + \beta_{21} X_1) X_2 \\
\dot{N}_2 &= N_2(\varepsilon_2 - \sigma_2 N_2 - \alpha_2 N_1)
\end{aligned} \tag{3.3}$$

3.1 *SIS* host-pathogen system

Restricting system (3.3) to a single host species, one obtains the following SIS model:

$$\begin{aligned} \dot{X} &= (\beta - \gamma - \varepsilon - \beta X)\, X \\ \dot{N} &= N(\varepsilon - \sigma N) \end{aligned} \tag{3.4}$$

where β is the contact rate and γ is the recovery rate. System (3.4) has three equilibrium points $E_0 = (0,0)$, $E_1 = \left(0, \frac{\varepsilon}{\sigma}\right)$ and $E_2 = \left(1 - \frac{1}{R_{10}}, \frac{\varepsilon}{\sigma}\right)$ where $R_{10} = \dfrac{\beta}{\gamma + \varepsilon}$ is the reproduction number of the infection. The two boundary equilibria $E_0 = (0,0)\,, E_1 = \left(0, \frac{\varepsilon}{\sigma}\right)$ always exist and $E_2 = \left(1 - \dfrac{1}{R_{10}}, \dfrac{\varepsilon}{\sigma}\right)$ exist if $R_{10} > 1$.

The following lemma can be easily proved

Lemma 2 *System (3.4) always has the equilibria E_0 and E_1. E_0 is always unstable. As for the existence of an internal equilibrium and the stability of them all, we have two cases:*

1. *If $R_{10} < 1$, the diseas-free equilibrium E_1 is globally asymptotically stable in the region $\{(X,N)\,|X \geqslant 0, N \geqslant 0\}$.*
2. *If $R_{10} > 1$, there exists a unique internal equilibrium E_2 which is globally asymptotically stable in the region $\{(X,N)\,|X \geqslant 0, N \geqslant .0\}$. In this case E_1 becomes unstable.*

3.2 Equilibrium Points Of System (3.3)

Let

$$R_{11} = \frac{\beta_{11}}{\gamma_1 + \varepsilon_1}, R_{12} = \frac{\beta_{22}}{\gamma_2 + \varepsilon_2}, R_{13} = \frac{\beta_{12}}{\gamma_1 + \varepsilon_1}, R_{14} = \frac{\beta_{21}}{\gamma_2 + \varepsilon_2}$$

System (3.3) has the following equilibria: $E_0 = (0,0,0,0)$, $E_1 = \left(0, \frac{\varepsilon_1}{\sigma_1}, 0, 0\right)$, $E_2 = \left(0, 0, 0, \frac{\varepsilon_2}{\sigma_2}\right)$ and $E_3 = (0, N_{1E}, 0, N_{2E})$ where

$$N_{1E} = \frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2} \quad , \quad N_{2E} = \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}$$

Finally we may find an internal equilibrium $E_4 = (X_{1E}, N_{1E}, X_{2E}, N_{2E})$ where X_{1E}, X_{2E} are the positive root of the following equations

$$\begin{aligned}\beta_{12}X_2-(\gamma_1+\varepsilon_1-\beta_{11}+\beta_{11}X_1+\beta_{12}X_2)\,X_1 &= 0\\ \beta_{21}X_1-(\gamma_2+\varepsilon_2-\beta_{22}+\beta_{22}X_2+\beta_{21}X_1)\,X_2 &= 0\end{aligned}$$

System (3.3) always has the following three boundary equilibria E_0, E_1 and E_2. E_3 exist if $\frac{\varepsilon_1}{\alpha_1}>\frac{\varepsilon_2}{\sigma_2}$, $\frac{\varepsilon_2}{\alpha_2}>\frac{\varepsilon_1}{\sigma_1}$and $\frac{\sigma_1}{\alpha_1}>\frac{\alpha_2}{\sigma_2}$ (or all inequalities are reversed).

3.3 Local stability of equilibria

We used the Jacobian matrix and Routh-Hurwitz criteria to get stability conditions.

Let $A_{11}=\gamma_1+\gamma_2+\varepsilon_1+\varepsilon_2-\beta_{11}-\beta_{22}+\sigma_1N_{1E}+\sigma_2N_{2E}$

$$\begin{aligned}A_{21}=\;&\gamma_1\gamma_2+\gamma_1\varepsilon_2+\gamma_2\varepsilon_1+\varepsilon_1\varepsilon_2-\gamma_2\beta_{11}-\gamma_1\beta_{22}-\varepsilon_2\beta_{11}-\varepsilon_1\beta_{22}+\beta_{11}\beta_{22}\\ &-\beta_{12}\beta_{21}+\sigma_2\gamma_1N_{2E}+\sigma_2\gamma_2N_{2E}+\sigma_2\varepsilon_1N_{2E}+\sigma_2\varepsilon_2N_{2E}-\sigma_2\beta_{11}N_{2E}\\ &-\sigma_2\beta_{22}N_{2E}+\sigma_1\gamma_1N_{1E}+\sigma_1\gamma_2N_{1E}+\sigma_1\varepsilon_1N_{1E}+\sigma_1\varepsilon_2N_{1E}-\sigma_1\beta_{11}N_{1E}\\ &-\sigma_1\beta_{22}N_{1E}-\alpha_1\alpha_2N_{1E}N_{2E}+\sigma_1\sigma_2N_{1E}N_{2E}\end{aligned}$$

$$\begin{aligned}A_{31}=\;&\sigma_1\gamma_1\gamma_2N_{1E}+\sigma_1\gamma_1\varepsilon_2N_{1E}+\sigma_1\gamma_2\varepsilon_1N_{1E}+\sigma_1\varepsilon_1\varepsilon_2N_{1E}-\sigma_1\gamma_2\beta_{11}N_{1E}\\ &-\sigma_1\gamma_1\beta_{22}N_{1E}-\sigma_1\varepsilon_2\beta_{11}N_{1E}-\sigma_1\varepsilon_1\beta_{22}N_{1E}+\sigma_1\beta_{11}\beta_{22}N_{1E}\\ &-\sigma_1\beta_{12}\beta_{21}N_{1E}+\sigma_2\gamma_1\gamma_2N_{2E}+\sigma_2\gamma_1\varepsilon_2N_{2E}+\sigma_2\gamma_2\varepsilon_1N_{2E}\\ &+\sigma_2\varepsilon_1\varepsilon_2N_{2E}-\sigma_2\gamma_2\beta_{11}N_{2E}-\sigma_2\gamma_1\beta_{22}N_{2E}-\sigma_2\varepsilon_2\beta_{11}N_{2E}\\ &-\sigma_2\varepsilon_1\beta_{22}N_{2E}+\sigma_2\beta_{11}\beta_{22}N_{2E}-\sigma_2\beta_{12}\beta_{21}N_{2E}-\alpha_1\alpha_2\gamma_1N_{1E}N_{2E}\\ &+\sigma_1\sigma_2\gamma_1N_{1E}N_{2E}-\alpha_1\alpha_2\gamma_2N_{1E}N_{2E}+\sigma_1\sigma_2\gamma_2N_{1E}N_{2E}-\alpha_1\alpha_2\varepsilon_1N_{1E}N_{2E}\\ &+\sigma_1\sigma_2\varepsilon_1N_{1E}N_{2E}-\alpha_1\alpha_2\varepsilon_2N_{1E}N_{2E}+\sigma_1\sigma_2\varepsilon_2N_{1E}N_{2E}+\alpha_1\alpha_2\beta_{11}N_{1E}N_{2E}\\ &-\sigma_1\sigma_2\beta_{11}N_{1E}N_{2E}+\alpha_1\alpha_2\beta_{22}N_{1E}N_{2E}-\sigma_1\sigma_2\beta_{22}N_{1E}N_{2E}\end{aligned}$$

$$\begin{aligned}A_{41}=\;&-\alpha_1\alpha_2\gamma_1\gamma_2N_{1E}N_{2E}+\sigma_1\sigma_2\gamma_1\gamma_2N_{1E}N_{2E}-\alpha_1\alpha_2\gamma_1\varepsilon_2N_{1E}N_{2E}\\ &-\alpha_1\alpha_2\gamma_2\varepsilon_1N_{1E}N_{2E}+\sigma_1\sigma_2\gamma_1\varepsilon_2N_{1E}N_{2E}+\sigma_1\sigma_2\gamma_2\varepsilon_1N_{1E}N_{2E}\\ &-\alpha_1\alpha_2\varepsilon_1\varepsilon_2N_{1E}N_{2E}+\sigma_1\sigma_2\varepsilon_1\varepsilon_2N_{1E}N_{2E}+\alpha_1\alpha_2\gamma_2\beta_{11}N_{1E}N_{2E}\\ &-\sigma_1\sigma_2\gamma_2\beta_{11}N_{1E}N_{2E}+\alpha_1\alpha_2\gamma_1\beta_{22}N_{1E}N_{2E}-\sigma_1\sigma_2\gamma_1\beta_{22}N_{1E}N_{2E}\\ &+\alpha_1\alpha_2\varepsilon_2\beta_{11}N_{1E}N_{2E}-\sigma_1\sigma_2\varepsilon_2\beta_{11}N_{1E}N_{2E}+\alpha_1\alpha_2\varepsilon_1\beta_{22}N_{1E}N_{2E}\\ &-\sigma_1\sigma_2\varepsilon_1\beta_{22}N_{1E}N_{2E}-\alpha_1\alpha_2\beta_{11}\beta_{22}N_{1E}N_{2E}+\alpha_1\alpha_2\beta_{12}\beta_{21}N_{1E}N_{2E}\\ &+\sigma_1\sigma_2\beta_{11}\beta_{22}N_{1E}N_{2E}-\sigma_1\sigma_2\beta_{12}\beta_{21}N_{1E}N_{2E}\end{aligned}$$

Also let

$a_1=\beta_{11}-\gamma_1-\varepsilon_1-2\beta_{11}X_{1E}-\beta_{12}X_{2E}$

$a_2=\beta_{12}\,(1-X_{1E})$

$a_3=-\sigma_1N_{1E}$

$a_4=-\alpha_1N_{1E}$

$a_5=\beta_{21}\,(1-X_{2E})$

$a_6=\beta_{22}-\gamma_2-\varepsilon_2-2\beta_{22}X_{2E}-\beta_{21}X_{1E}$

$a_7=-\alpha_2N_{2E}$

$a_8=-\sigma_2N_{2E}$

and
$A_{12} = -(a_1 + a_3 + a_6 + a_8)$
$A_{22} = a_1a_3 + a_1a_6 - a_2a_5 + a_1a_8 + a_3a_6 + a_3a_8 - a_4a_7 + a_6a_8$
$A_{32} = -a_1a_3a_6 + a_2a_3a_5 - a_1a_3a_8 + a_1a_4a_7 - a_1a_6a_8 + a_2a_5a_8 - a_3a_6a_8 + a_4a_6a_7$
$A_{42} = a_1a_3a_6a_8 - a_1a_4a_6a_7 - a_2a_3a_5a_8 + a_2a_4a_5a_7$

The results of this section are summarized in the following Theorem

Theorem 3 *System (3.3) always has the boundary equilibria* E_0, E_1 *and* E_2. E_0 *is always unstable. As for the existence of the boundary equilibrium* E_3 *and an internal equilibrium and the stability of them all, we have the following cases:*

1. *If* $\frac{\varepsilon_1}{\sigma_1} < \frac{\varepsilon_2}{\alpha_2}$ *,* $\frac{\varepsilon_2}{\sigma_2} < \frac{\varepsilon_1}{\alpha_1}$ *and* $\frac{\sigma_1}{\alpha_1} > \frac{\alpha_2}{\sigma_2}$*, there exists another boundary equilibrium* E_3. E_3 *is locally asymptotically stable if and only if* $A_{11} > 0, A_{31} > 0, A_{41} > 0$ *and* $A_{11}A_{21}A_{31} > A_{31}^2 + A_{11}^2 A_{41}$. *In this case* E_1 *and* E_2 *are unstable.*

2. *If* $\frac{\varepsilon_1}{\sigma_1} > \frac{\varepsilon_2}{\alpha_2}$ *,* $\frac{\varepsilon_2}{\sigma_2} > \frac{\varepsilon_1}{\alpha_1}$ *and* $\frac{\sigma_1}{\alpha_1} < \frac{\alpha_2}{\sigma_2}$*, the boundary equilibrium* E_3 *also exists, which is locally asymptotically stable if and only if* $A_{11} > 0, A_{31} > 0, A_{41} > 0$ *and* $A_{11}A_{21}A_{31} > A_{31}^2 + A_{11}^2 A_{41}$. *In this case* E_1 *and* E_2 *are locally asymptotically stable if and only if* $R_{11} < 1$*,*$R_{12} < 1$*,*$(R_{11} - 1)(R_{12} - 1) > R_{13}R_{14}$.

3. *If* $\frac{\varepsilon_1}{\sigma_1} > \frac{\varepsilon_2}{\alpha_2}$ *and* $\frac{\varepsilon_2}{\sigma_2} < \frac{\varepsilon_1}{\alpha_1}$*, the equilibrium* E_3 *does not exist,* E_1*is locally asymptotically stable if and only if* $R_{11} < 1$*,*$R_{12} < 1$*,*$(R_{11} - 1)(R_{12} - 1) > R_{13}R_{14}$. *In this case* E_2 *is unstable.*

4. *If* $\frac{\varepsilon_1}{\sigma_1} < \frac{\varepsilon_2}{\alpha_2}$ *and* $\frac{\varepsilon_2}{\sigma_2} > \frac{\varepsilon_1}{\alpha_1}$*, also the equilibrium* E_3 *does not exist,* E_2*is locally asymptotically stable if and only if* $R_{11} < 1$*,*$R_{12} < 1$*,*$(R_{11} - 1)(R_{12} - 1) > R_{13}R_{14}$. *In this case* E_1 *is unstable.*

5. *There may or may not exists an internal equilibrium* E_4. *If* E_4 *exist, then it is locally asymptotically stable if and only if* $A_{12} > 0, A_{32} > 0, A_{42} > 0$ *and* $A_{12}A_{22}A_{32} > A_{32}^2 + A_{12}^2 A_{42}$.

3.4 Bifurcation Analysis [4]

In this section, we discuss Hopf bifurcation theory for system (3.1). The System comprises six equations, but only four are necessary, since $N_i = I_i + S_i$.We choose to use as variables N_i and I_i for $i = 1, 2$, obtaining the following 4-dimensional system:

$$\begin{aligned}
\frac{dI_1}{dt} &= \frac{\beta_{12} I_2}{N_2} N_1 - \left(\gamma_1 - \beta_{11} + \sigma_1 N_1 + \alpha_1 N_2 + \frac{\beta_{11} I_1}{N_1} + \frac{\beta_{12} I_2}{N_2}\right) I_1 \\
\frac{dN_1}{dt} &= N_1(\varepsilon_1 - \sigma_1 N_1 - \alpha_1 N_2) \\
\frac{dI_2}{dt} &= \frac{\beta_{21} I_1}{N_1} N_2 - \left(\gamma_2 - \beta_{22} + \sigma_2 N_2 + \alpha_2 N_1 + \frac{\beta_{22} I_2}{N_2} + \frac{\beta_{21} I_1}{N_1}\right) I_2 \\
\frac{dN_2}{dt} &= N_2(\varepsilon_2 - \sigma_2 N_2 - \alpha_2 N_1)
\end{aligned}$$

To use the bifurcation theorem for this system we need to discuss Hopf bifurcation at an internal equilibrium. However, there are no explicit formula for an internal equilibrium and, in general, not even its existence can be proved. Hence we will study the bifurcation in the very special case, where all the analogous parameters are the same for species 1 and 2. Although it is a very particular case, it displays several interesting behaviors that can shed light also outside of this special structure.

Namely, we let

$$\begin{aligned}
\beta_{11} &= \beta_{22} = \beta_1, \beta_{12} = \beta_{21} = \beta_2, \gamma_1 = \gamma_2 = \gamma \\
\varepsilon_1 &= \varepsilon_2 = \varepsilon, \alpha_1 = \alpha_2 = \alpha, \sigma_1 = \sigma_2 = \sigma
\end{aligned}$$

Then the system becomes a symmetrical (with respect to the exchange of 1 and 2) system.

$$\begin{aligned}
\frac{dI_1}{dt} &= \frac{\beta_2 I_2}{N_2} N_1 - \left(\gamma - \beta_1 + \sigma N_1 + \alpha N_2 + \frac{\beta_1 I_1}{N_1} + \frac{\beta_2 I_2}{N_2}\right) I_1 \\
\frac{dN_1}{dt} &= N_1(\varepsilon - \sigma N_1 - \alpha N_2) \\
\frac{dI_2}{dt} &= \frac{\beta_2 I_1}{N_1} N_2 - \left(\gamma - \beta_1 + \sigma N_2 + \alpha N_1 + \frac{\beta_1 I_2}{N_2} + \frac{\beta_2 I_1}{N_1}\right) I_2 \\
\frac{dN_2}{dt} &= N_2(\varepsilon - \sigma N_2 - \alpha N_1)
\end{aligned} \tag{3.5}$$

The internal equilibrium point is $E^* = (I^*, N^*, I^*, N^*)$ where $N^* = \dfrac{\varepsilon}{\sigma + \alpha}$, $I^* = N^* \left(1 - \dfrac{1}{R^*}\right)$ and $R^* = \dfrac{(\beta_1 + \beta_2)}{\gamma + \varepsilon}$. E^* exists if $R^* > 1$.

The Jacobian matrix of system (3.5) at E^* is given by

$$J^* = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 0 & e_5 & 0 & e_6 \\ e_3 & e_4 & e_1 & e_2 \\ 0 & e_6 & 0 & e_5 \end{bmatrix}$$

Where

$e_1 = \dfrac{\beta_1}{R^*} - (\beta_1 + \beta_2)$

$e_2 = \left(\beta_1 \left(1 - \dfrac{1}{R^*}\right) + \beta_2\right)\left(1 - \dfrac{1}{R^*}\right) - \sigma I^*$

$e_3 = \dfrac{\beta_2}{R^*}$

$e_4 = -\alpha I^* - \dfrac{\beta_2}{R^*}\left(1 - \dfrac{1}{R^*}\right)$

$e_5 = -\sigma N^*$

$e_6 = -\alpha N^*$

In the following, we choose the interinfection rate of disease between the two species β_2 as the bifurcation parameter, and fix the other parameters.

Theorem 4 *Assume that $E^* = (I^*, N^*, I^*, N^*)$ exists and $e_1 < 0$, $e_5^2 > e_6^2$ and $e_1^2 > e_3^2$, then there is a positive number β_2^* such that system (3.5) may exhibit a Hopf bifurcation leading to a family of periodic solutions that bifurcates from the equilibrium point E^* for suitable values of β_2 in a neighborhood of β_2^*.*

Proof. The eigenvalues of J^* satisfy

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \tag{3.6}$$

where

$$\begin{aligned} A_1 &= -2(e_1 + e_5) \\ A_2 &= e_1^2 + e_5^2 + 4e_1e_5 - e_3^2 - e_6^2 \\ A_3 &= -2e_1^2e_5 - 2e_1e_5^2 + 2e_1e_6^2 + 2e_3^2e_5 \\ A_4 &= e_1^2e_5^2 - e_1^2e_6^2 - e_3^2e_5^2 + e_3^2e_6^2 \end{aligned}$$

Assume that $e_1 < 0$, $e_5^2 > e_6^2$ and $e_1^2 > e_3^2$.

$e_5 = -\sigma N^* < 0$ always. Then, we have

$$\begin{aligned} A_1 &= -2(e_1 + e_5) > 0 \\ A_3 &= -2e_1^2e_5 - 2e_1e_5^2 + 2e_1e_6^2 + 2e_3^2e_5 = -2e_5\left(e_1^2 - e_3^2\right) - 2e_1\left(e_5^2 - e_6^2\right) > 0 \\ A_4 &= e_1^2e_5^2 - e_1^2e_6^2 - e_3^2e_5^2 + e_3^2e_6^2 = e_1^2\left(e_5^2 - e_6^2\right) - e_3^2\left(e_5^2 - e_6^2\right) \\ &= \left(e_5^2 - e_6^2\right)\left(e_1^2 - e_3^2\right) > 0 \end{aligned}$$

By Routh-Hurwitz criteria a necessary and sufficient conditions for the eigenvalues to have a negative real part are:

$$A_1 > 0, A_3 > 0, A_4 > 0 \text{ and } A_1A_2A_3 > A_3^2 + A_1^2A_4$$

From the above we have $A_1 > 0$, $A_3 > 0$ and $A_4 > 0$.So,equation (3.6) will have two pure imaginary roots if and only if

$$A_1A_2A_3 = A_3^2 + A_1^2A_4$$

for some values of β_2, say $\beta_2 = \beta_2^*$.Since at $\beta_2 = \beta_2^*$ there is an interval containing β_2^*, say $(\beta_2^*-\varepsilon, \beta_2^*+\varepsilon)$ for some $\varepsilon > 0$ for which $\beta_2 \in (\beta_2^*-\varepsilon, \beta_2^*+\varepsilon)$. Thus for $\beta_2 \in (\beta_2^* - \varepsilon, \beta_2^* + \varepsilon)$, equation (3.6) cannot have real positive roots. For $\beta_2 = \beta_2^*$,equation (3.6) can be factored into the form

$$(\lambda^2 + \theta_1)(\lambda + \theta_2)(\lambda + \theta_3) = 0, \theta_i > 0, i = 1, 2, 3.$$

where

$$\begin{aligned} A_1 &= \theta_2 + \theta_3 \\ A_2 &= \theta_1 + \theta_2\theta_3 \\ A_3 &= \theta_1 (\theta_2 + \theta_3) \\ A_4 &= \theta_1\theta_2\theta_3 \end{aligned}$$

In particular, the set of roots of (3.6) is given by $P(\beta_2) = \left\{i\sqrt{\theta_1}, -i\sqrt{\theta_1}, -\theta_2, -\theta_3\right\}$ .Then equation (3.6) has two pure imaginary roots for some value of β_2 say $\beta_2 = \beta_2^*$. But for $\beta_2 \in (\beta_2^* - \varepsilon, \beta_2^* + \varepsilon)$ the roots are in general form

$$\begin{aligned} \lambda_1(\beta_2) &= \nu(\beta_2) + i\mu(\beta_2) \\ \lambda_2(\beta_2) &= \nu(\beta_2) - i\mu(\beta_2) \\ \lambda_3(\beta_2) &= -\theta_2 \\ \lambda_4(\beta_2) &= -\theta_3 \end{aligned}$$

First, substituting $\lambda_i(\beta_2)$, $i = 1, 2$ into equation (3.6) we get the equations

$$\begin{aligned} \mu^4 - 6\mu^2\nu^2 - 3A_1\mu^2\nu - A_2\mu^2 + \nu^4 + A_1\nu^3 + A_2\nu^2 + A_3\nu + A_4 &= 0 \\ -4\mu^3\nu - A_1\mu^3 + 4\mu\nu^3 + 3A_1\mu\nu^2 + 2A_2\mu\nu + A_3\mu &= 0 \quad (3.7) \end{aligned}$$

Differentiating (3.7) with respect to β_2, we get

$$\begin{aligned} A(\beta_2)\nu'(\beta_2) - B(\beta_2)\mu'(\beta_2) + C(\beta_2) &= 0 \\ B(\beta_2)\nu'(\beta_2) + A(\beta_2)\mu'(\beta_2) + D(\beta_2) &= 0 \end{aligned}$$

where

$$\begin{aligned}
A(\beta_2) &= 4\nu\left(\nu^2-\mu^2\right)-8\nu\mu^2+3A_1\left(\nu^2-\mu^2\right)+2\nu A_2+A_3\\
B(\beta_2) &= 4\mu\left(\nu^2-\mu^2\right)+8\nu^2\mu+6\nu\mu A_1+2\mu A_2\\
C(\beta_2) &= A_1'\left[\nu\left(\nu^2-\mu^2\right)-2\nu\mu^2\right]+A_2'\left(\nu^2-\mu^2\right)+\nu A_3'+A_4'\\
D(\beta_2) &= A_1'\left[\mu\left(\nu^2-\mu^2\right)+2\nu^2\mu\right]+2\nu\mu A_2'+\mu A_3'
\end{aligned}$$

Since $A(\beta_2^*)C(\beta_2^*)+B(\beta_2^*)D(\beta_2^*)\neq 0$, we have

$$\mathrm{Re}\left[\frac{d\lambda_i}{d\beta_2}\right]_{\beta_2=\beta_2^*}=\nu'(\beta_2^*)=-\frac{A(\beta_2)C(\beta_2)+B(\beta_2)D(\beta_2)}{A^2(\beta_2)+B^2(\beta_2)}\Bigg|_{\beta_2=\beta_2^*}\neq 0$$

Therefore, we can apply Hopf bifurcation theorem [3] to prove that system (3.5) exhibits a Hopf bifurcation at E^* leading to a family of periodic solutions that bifurcates from the equilibrium point E^* for some $\beta_2\in(\beta_2^*-\varepsilon,\beta_2^*+\varepsilon)$. This completes the proof. ■

4 Competition *SIS* Model with Mass Action Incidence

Consider the following autonomous competition SIS model, which is similar to the model in the previous section, but here we use the mass action incidence instead of the standard incidence:

$$\begin{aligned}
\frac{dI_1}{dt} &= \left(\beta_{11}I_1+\beta_{12}I_2\right)S_1-\left(\gamma_1+\sigma_1N_1+\alpha_1N_2\right)I_1\\
\frac{dS_1}{dt} &= \varepsilon_1N_1+\gamma_1I_1-\left(\sigma_1N_1+\alpha_1N_2+\beta_{11}I_1+\beta_{12}I_2\right)S_1\\
\frac{dN_1}{dt} &= N_1(\varepsilon_1-\sigma_1N_1-\alpha_1N_2) \qquad (4.1)\\
\frac{dI_2}{dt} &= \left(\beta_{22}I_2+\beta_{21}I_1\right)S_2-\left(\gamma_2+\sigma_2N_2+\alpha_2N_1\right)I_2\\
\frac{dS_2}{dt} &= \varepsilon_2N_2+\gamma_2I_2-\left(\sigma_2N_2+\alpha_2N_1+\beta_{22}I_2+\beta_{21}I_1\right)S_2\\
\frac{dN_2}{dt} &= N_2(\varepsilon_2-\sigma_2N_2-\alpha_2N_1)
\end{aligned}$$

System (4.1) comprises six equations, but only four are necessary, since $N_i=I_i+S_i$.We choose to use as variables N_i and I_i for $i=1,2$, obtaining the following 4-dimensional system:

$$
\begin{aligned}
\frac{dI_1}{dt} &= \beta_{12}I_2N_1 - (\gamma_1 - \beta_{11}N_1 + \sigma_1 N_1 + \alpha_1 N_2 + \beta_{11}I_1 + \beta_{12}I_2)\, I_1 \\
\frac{dN_1}{dt} &= N_1(\varepsilon_1 - \sigma_1 N_1 - \alpha_1 N_2) \\
\frac{dI_2}{dt} &= \beta_{21}I_1N_2 - (\gamma_2 - \beta_{22}N_2 + \sigma_2 N_2 + \alpha_2 N_1 + \beta_{22}I_2 + \beta_{21}I_1)\, I_2 \\
\frac{dN_2}{dt} &= N_2(\varepsilon_2 - \sigma_2 N_2 - \alpha_2 N_1)
\end{aligned} \tag{4.2}
$$

Before proceeding, we briefly summarize known results on the host-pathogen model.

4.1 *SIS* host-pathogen system

Restricting system (4.2) to a single host species, one obtains the following SIS model:

$$
\begin{aligned}
\frac{dI}{dt} &= (\beta N - \beta I - \gamma - \sigma N)\, I \\
\frac{dN}{dt} &= N(\varepsilon - \sigma N)
\end{aligned} \tag{4.3}
$$

where β is the contact rate and γ is the recovery rate.

System (4.3) has three equilibrium points $E_0 = (0,0)$, $E_1 = \left(0, \frac{\varepsilon}{\sigma}\right)$ and $E_2 = \left(\frac{\varepsilon}{\sigma}\left(1 - \frac{1}{R_{20}}\right), \frac{\varepsilon}{\sigma}\right)$ where $R_{20} = \dfrac{\beta\frac{\varepsilon}{\sigma}}{\gamma + \varepsilon}$ is the reproduction number of the infection. Note that $R_{20} = \dfrac{\varepsilon}{\sigma}R_{10}$. System (4.3) always has the following two boundary equilibria E_0, E_1. E_2 exist if $R_{20} > 1$

The following lemma can be easily proved

Lemma 5 *System (4.3) always has the equilibria E_0 and E_1. E_0 is always unstable. As for the existence of an internal equilibrium and the stability of them all, we have two cases:*

1. *If $R_{20} < 1$, the diseas-free equilibrium E_1 is globally asymptotically stable in the region $\{(I,N)\,|I \geqslant 0, N \geqslant 0\}$.*

2. *If $R_{20} > 1$, there exists a unique internal equilibrium E_2 which is globally asymptotically stable in the region $\{(I,N)\,|I \geqslant 0, N \geqslant 0\}$. In this case E_1 becomes unstable.*

4.2 Equilibrium Points of System (4.2)

Let

$$R_{21} = \frac{\beta_{11}\frac{\varepsilon_1}{\sigma_1}}{\gamma_1 + \varepsilon_1}, R_{22} = \frac{\beta_{22}\frac{\varepsilon_2}{\sigma_2}}{\gamma_2 + \varepsilon_2}$$

System (4.2) has the following equilibria: $E_0 = (0,0,0,0)$, $E_1 = \left(0, \frac{\varepsilon_1}{\sigma_1}, 0, 0\right)$, $E_2 = \left(\frac{\varepsilon_1}{\sigma_1}\ \left(1 - \frac{1}{R_{21}}\right), \frac{\varepsilon_1}{\sigma_1}, 0, 0\right)$, $E_3 = \left(0,0,0,\frac{\varepsilon_2}{\sigma_2}\right)$, $E_4 = \left(0, 0, \frac{\varepsilon_2}{\sigma_2}\ \left(1 - \frac{1}{R_{22}}\right), \frac{\varepsilon_2}{\sigma_2}\right)$ and $E_5 = (0, N_{1E}, 0, N_{2E})$ where

$$N_{1E} = \frac{\varepsilon_1\sigma_2 - \varepsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2} \quad , \quad N_{2E} = \frac{\varepsilon_2\sigma_1 - \varepsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}$$

Finally, we may find an internal equilibrium $E_6 = (I_{1E}, N_{1E}, I_{2E}, N_{2E})$ where I_{1E}, I_{2E} are the positive roots of the following equations

$$\begin{aligned} \beta_{12}I_2N_{1E} - (\gamma_1 - \beta_{11}N_{1E} + \varepsilon_1 + \beta_{11}I_1 + \beta_{12}I_2)\, I_1 &= 0 \\ \beta_{21}I_1N_{2E} - (\gamma_2 - \beta_{22}N_{2E} + \varepsilon_2 + \beta_{22}I_2 + \beta_{21}I_1)\, I_2 &= 0 \end{aligned}$$

System (4.2) always has the three boundary equilibria E_0, E_1, and E_3. E_2 exist if $R_{21} > 1$, E_4 exist if $R_{22} > 1$ and E_5 exist if $\frac{\varepsilon_1}{\alpha_1} > \frac{\varepsilon_2}{\sigma_2}$, $\frac{\varepsilon_2}{\alpha_2} > \frac{\varepsilon_1}{\sigma_1}$and $\frac{\sigma_1}{\alpha_1} > \frac{\alpha_2}{\sigma_2}$ (or all inequalities are reversed).

4.3 Local stability of the equilibria

As we did in section 3 we used Routh-Hurwitz criteria to get the following result:

Let

$$A_{13} = \gamma_1 + \gamma_2 + \varepsilon_1 + \varepsilon_2 + \sigma_1 N_E - \beta_{11}N_{1E} + \sigma_2 N_{2E} - \beta_{22}N_{2E}$$

$$\begin{aligned} A_{23} = {} & \gamma_1\gamma_2 + \gamma_1\varepsilon_2 + \gamma_2\varepsilon_1 + \varepsilon_1\varepsilon_2 + \sigma_2\gamma_1 N_{2E} + \sigma_2\gamma_2 N_{2E} + \sigma_2\varepsilon_1 N_{2E} + \sigma_2\varepsilon_2 N_{2E} \\ & - \gamma_1\beta_{22}N_{2E} - \varepsilon_1\beta_{22}N_{2E} - \sigma_1\beta_{11}N_E^2 - \sigma_2\beta_{22}N_{2E}^2 + \sigma_1\gamma_1 N_{1E} + \sigma_1\gamma_2 N_{1E} \\ & + \sigma_1\varepsilon_1 N_{1E} + \sigma_1\varepsilon_2 N_{1E} - \gamma_2\beta_{11}N_{1E} - \varepsilon_2\beta_{11}N_{1E} - \alpha_1\alpha_2 N_{1E}N_{2E} + \sigma_1\sigma_2 N_{1E}N_{2E} \\ & - \sigma_2\beta_{11}N_{1E}N_{2E} - \sigma_1\beta_{22}N_{1E}N_{2E} + \beta_{11}\beta_{22}N_{1E}N_{2E} - \beta_{12}\beta_{21}N_{1E}N_{2E} \end{aligned}$$

$$\begin{aligned} A_{33} = {} & -\sigma_2\gamma_1\beta_{22}N_{2E}^2 - \sigma_2\varepsilon_1\beta_{22}N_{2E}^2 + \sigma_1\gamma_1\gamma_2 N_{1E} + \sigma_1\gamma_1\varepsilon_2 N_{1E} + \sigma_1\gamma_2\varepsilon_1 N_{1E} \\ & + \sigma_1\varepsilon_1\varepsilon_2 N_{1E} + \sigma_2\gamma_1\gamma_2 N_{2E} + \sigma_2\gamma_1\varepsilon_2 N_{2E} + \sigma_2\gamma_2\varepsilon_1 N_{2E} + \sigma_2\varepsilon_1\varepsilon_2 N_{2E} \\ & - \sigma_1\gamma_2\beta_{11}N_{1E}^2 - \sigma_1\varepsilon_2\beta_{11}N_{1E}^2 - \alpha_1\alpha_2\gamma_1 N_{1E}N_{2E} + \sigma_1\sigma_2\gamma_1 N_{1E}N_{2E} \\ & - \alpha_1\alpha_2\gamma_2 N_{1E}N_{2E} + \sigma_1\sigma_2\gamma_2 N_{1E}N_{2E} - \alpha_1\alpha_2\varepsilon_1 N_{1E}N_{2E} + \sigma_1\sigma_2\varepsilon_1 N_{1E}N_{2E} \\ & - \alpha_1\alpha_2\varepsilon_2 N_{1E}N_{2E} + \sigma_1\sigma_2\varepsilon_2 N_{1E}N_{2E} - \sigma_1\gamma_1\beta_{22}N_{1E}N_{2E} - \sigma_2\gamma_2\beta_{11}N_{1E}N_{2E} \\ & - \sigma_1\varepsilon_1\beta_{22}N_{1E}N_{2E} - \sigma_2\varepsilon_2\beta_{11}N_{1E}N_{2E} + \alpha_1\alpha_2\beta_{11}N_{1E}^2N_{2E} - \sigma_1\sigma_2\beta_{11}N_{1E}^2N_{2E} \\ & + \alpha_1\alpha_2\beta_{22}N_{1E}N_{2E}^2 - \sigma_1\sigma_2\beta_{22}N_{1E}N_{2E}^2 + \sigma_1\beta_{11}\beta_{22}N_{1E}^2N_{2E} \end{aligned}$$

$$
\begin{aligned}
&-\sigma_1\beta_{12}\beta_{21}N_{1E}^2N_{2E} + \sigma_2\beta_{11}\beta_{22}N_{1E}N_{2E}^2 - \sigma_2\beta_{12}\beta_{21}N_{1E}N_{2E}^2 \\
A_{43} = &-\alpha_1\alpha_2\beta_{11}\beta_{22}N_{1E}^2N_{2E}^2 + \alpha_1\alpha_2\beta_{12}\beta_{21}N_{1E}^2N_{2E}^2 + \sigma_1\sigma_2\beta_{11}\beta_{22}N_{1E}^2N_{2E}^2 \\
&-\sigma_1\sigma_2\beta_{12}\beta_{21}N_{1E}^2N_{2E}^2 + \alpha_1\alpha_2\gamma_2\beta_{11}N_{1E}^2N_{2E} - \sigma_1\sigma_2\gamma_2\beta_{11}N_{1E}^2N_{2E} \\
&+\alpha_1\alpha_2\gamma_1\beta_{22}N_{1E}N_{2E}^2 - \sigma_1\sigma_2\gamma_1\beta_{22}N_{1E}N_{2E}^2 + \alpha_1\alpha_2\varepsilon_2\beta_{11}N_{1E}^2N_{2E} \\
&-\sigma_1\sigma_2\varepsilon_2\beta_{11}N_{1E}^2N_{2E} + \alpha_1\alpha_2\varepsilon_1\beta_{22}N_{1E}N_{2E}^2 - \sigma_1\sigma_2\varepsilon_1\beta_{22}N_{1E}N_{2E}^2 \\
&-\alpha_1\alpha_2\gamma_1\gamma_2N_{1E}N_{2E} + \sigma_1\sigma_2\gamma_1\gamma_2N_{1E}N_{2E} - \alpha_1\alpha_2\gamma_1\varepsilon_2N_{1E}N_{2E} \\
&-\alpha_1\alpha_2\gamma_2\varepsilon_1N_{1E}N_{2E} + \sigma_1\sigma_2\gamma_1\varepsilon_2N_{1E}N_{2E} + \sigma_1\sigma_2\gamma_2\varepsilon_1N_{1E}N_{2E} \\
&-\alpha_1\alpha_2\varepsilon_1\varepsilon_2N_{1E}N_{2E} + \sigma_1\sigma_2\varepsilon_1\varepsilon_2N_{1E}N_{2E}
\end{aligned}
$$

Let

$$
\begin{aligned}
b_1 &= \beta_{11}N_{1E} - 2\beta_{11}I_{1E} - \gamma_1 - \varepsilon_1 - \beta_{12}I_{2E} \\
b_2 &= (\beta_{11} - \sigma_1)\, I_{1E} + \beta_{12}I_{2E} \\
b_3 &= \beta_{12}(N_{1E} - I_{1E}) \\
b_4 &= -\alpha_1 I_{1E} \\
b_5 &= -\sigma_1 N_{1E} \\
b_6 &= -\alpha_1 N_{1E} \\
b_7 &= \beta_{21}(N_{2E} - I_{2E}) \\
b_8 &= -\alpha_2 I_{2E} \\
b_9 &= \beta_{22}N_{2E} - 2\beta_{22}I_{2E} - \gamma_2 - \varepsilon_2 - \beta_{21}I_{1E} \\
b_{10} &= (\beta_{22} - \sigma_2)\, I_{2E} + \beta_{21}I_{1E} \\
b_{11} &= -\alpha_2 N_{2E} \\
b_{12} &= -\sigma_2 N_{2E}
\end{aligned}
$$

Let

$$
\begin{aligned}
A_{14} &= -\left(b_1 + b_5 + b_9 + b_{12}\right) \\
A_{24} &= b_1b_5 + b_1b_9 - b_3b_7 + b_5b_9 + b_1b_{12} + b_5b_{12} - b_6b_{11} + b_9b_{12} \\
A_{34} &= -b_1b_5b_9 + b_3b_5b_7 - b_1b_5b_{12} + b_1b_6b_{11} - b_1b_9b_{12} + b_3b_7b_{12} - b_5b_9b_{12} + b_6b_9b_{11} \\
A_{44} &= b_1b_5b_9b_{12} - b_1b_6b_9b_{11} - b_3b_5b_7b_{12} + b_3b_6b_7b_{11}
\end{aligned}
$$

The results of this section are summarized in the following Theorem

Theorem 6 *System (4.2) always has the boundary equilibria* E_0, E_1 *and* E_3. E_0 *is always unstable. As for the existence of the other equilibria and the stability of them all, we have the following cases:*

1. *If* $R_{21} > 1$, *the equilibrium* E_2 *exists, which is locally asymptotically stable if and only if* $\dfrac{\varepsilon_2}{\alpha_2} < \dfrac{\varepsilon_1}{\sigma_1}$. *In this case* E_1 *is unstable.*

2. *If* $R_{22} > 1$, *the equilibrium* E_4 *exists, which is locally asymptotically stable if and only if* $\dfrac{\varepsilon_1}{\alpha_1} < \dfrac{\varepsilon_2}{\sigma_2}$. *In this case* E_3 *is unstable.*

3. *If* $\dfrac{\varepsilon_1}{\sigma_1} < \dfrac{\varepsilon_2}{\alpha_2}$, $\dfrac{\varepsilon_2}{\sigma_2} < \dfrac{\varepsilon_1}{\alpha_1}$ *and* $\dfrac{\sigma_1}{\alpha_1} > \dfrac{\alpha_2}{\sigma_2}$, *there exists another boundary equilibrium* E_5. E_5 *is locally asymptotically stable if and only if* $A_{13} > 0, A_{33} > 0, A_{43} > 0$ *and* $A_{13}A_{23}A_{33} > A_{33}^2 + A_{13}^2 A_{43}$. *In this case* E_1 *and* E_3 *are unstable.*

4. *If* $\dfrac{\varepsilon_1}{\sigma_1} > \dfrac{\varepsilon_2}{\alpha_2}$, $\dfrac{\varepsilon_2}{\sigma_2} > \dfrac{\varepsilon_1}{\alpha_1}$ *and* $\dfrac{\sigma_1}{\alpha_1} < \dfrac{\alpha_2}{\sigma_2}$, *the boundary equilibrium* E_5 *also exists, which is locally asymptotically stable if and only if* $A_{13} > 0, A_{33} > 0, A_{43} > 0$ *and* $A_{13}A_{23}A_{33} > A_{33}^2 + A_{13}^2 A_{43}$. *In this case* E_1 *is locally asymptotically stable if and only if* $R_{21} < 1$. *And* E_3 *is locally asymptotically stable if and only if* $R_{22} < 1$.

5. *If* $\dfrac{\varepsilon_1}{\sigma_1} > \dfrac{\varepsilon_2}{\alpha_2}$ *and* $\dfrac{\varepsilon_2}{\sigma_2} < \dfrac{\varepsilon_1}{\alpha_1}$, *the equilibrium* E_5 *does not exist,* E_1*is locally asymptotically stable if and only if* $R_{21} < 1$. *In this case* E_3 *is unstable.*

6. *If* $\dfrac{\varepsilon_1}{\sigma_1} < \dfrac{\varepsilon_2}{\alpha_2}$ *and* $\dfrac{\varepsilon_2}{\sigma_2} > \dfrac{\varepsilon_1}{\alpha_1}$, *also the equilibrium* E_5 *does not exist,* E_3*is locally asymptotically stable if and only if* $R_{22} < 1$. *In this case* E_1 *is unstable.*

7. *There may or may not exists an internal equilibrium* E_6. *If* E_6 *exist, then it is locally asymptotically stable if and only if* $A_{14} > 0, A_{34} > 0, A_{44} > 0$ *and* $A_{14}A_{24}A_{34} > A_{34}^2 + A_{14}^2 A_{44}$.

4.4 Bifurcation Analysis

To use Hopf bifurcation theorem for system (4.2) we need to discuss Hopf bifurcation at an internal equilibrium. Again we will study the bifurcation in the very special case where

$$\begin{aligned} \beta_{11} &= \beta_{22} = \beta_1, \beta_{12} = \beta_{21} = \beta_2, \gamma_1 = \gamma_2 = \gamma \\ \varepsilon_1 &= \varepsilon_2 = \varepsilon, \alpha_1 = \alpha_2 = \alpha, \sigma_1 = \sigma_2 = \sigma \end{aligned}$$

Then system (4.2) becomes a symmetrical (with respect to the exchange of 1 and 2) system.

$$\begin{aligned} \frac{dI_1}{dt} &= \beta_2 I_2 N_1 - (\gamma - \beta_1 N_1 + \sigma N_1 + \alpha N_2 + \beta_1 I_1 + \beta_2 I_2)\, I_1 \\ \frac{dN_1}{dt} &= N_1(\varepsilon - \sigma N_1 - \alpha N_2) \\ \frac{dI_2}{dt} &= \beta_2 I_1 N_2 - (\gamma - \beta_1 N_2 + \sigma N_2 + \alpha N_1 + \beta_1 I_2 + \beta_2 I_1)\, I_2 \\ \frac{dN_2}{dt} &= N_2(\varepsilon - \sigma N_2 - \alpha N_1) \end{aligned} \tag{4.4}$$

The internal equilibrium point is $E^* = (I^*, N^*, I^*, N^*)$ where $N^* = \dfrac{\varepsilon}{\sigma + \alpha}$, $I^* = N^*\left(1 - \dfrac{1}{R^*}\right)$ and $R^* = \dfrac{N^*(\beta_1 + \beta_2)}{\gamma + \varepsilon}$. $E^* = (I^*, N^*, I^*, N^*)$ exists if $R^* > 1$.

The Jacobian matrix of system (4.4) at E^* is given by

$$J^* = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ 0 & f_5 & 0 & f_6 \\ f_3 & f_4 & f_1 & f_2 \\ 0 & f_6 & 0 & f_5 \end{bmatrix}$$

where

$$f_1 = \left(-\beta_1 - \beta_2 + \frac{\beta_1}{R^*}\right) N^*$$
$$f_2 = (\beta_1 + \beta_2 - \sigma) I^*$$
$$f_3 = \frac{\beta_2 N^*}{R^*}$$
$$f_4 = -\alpha I^*$$
$$f_5 = -\sigma N^*$$
$$f_6 = -\alpha N^*$$

As we did in the bifurcation analysis of SIS with standard incidence, we can obtain a similar result for the existence of a Hopf bifurcation for the parameter β_2. The following theorem is similar to theorem 2, so we will only state the theorem and the proof will be omitted.

Theorem 7 *Assume that $E^* = (I^*, N^*, I^*, N^*)$ exists and $f_1 < 0, f_5^2 > f_6^2$ and $f_1^2 > f_3^2$, then there is a positive number β_2^* such that system (4.4) may exhibit a Hopf bifurcation leading to a family of periodic solutions that bifurcates from the equilibrium point E^* for suitable values of β_2 in a neighborhood of β_2^*.*

5 CONCLUSION

In this section we summarize the previous results and give some biological interpretations:

Competition *SIS* Model with Standard Incidence

Equilibrium	Existence?	Stable?
$E_0 = (0,0,0,0)$	Yes	No
$E_1 = \left(0, \frac{\varepsilon_1}{\sigma_1}, 0, 0\right)$	Yes	$R_{11} < 1, R_{12} < 1, (R_{11} - 1)(R_{12} - 1) > R_{13}R_{14}$ and $\frac{\varepsilon_2}{\alpha_2} < \frac{\varepsilon_1}{\sigma_1}$
$E_2 = \left(0, 0, 0, \frac{\varepsilon_2}{\sigma_2}\right)$	Yes	$R_{11} < 1, R_{12} < 1, (R_{11} - 1)(R_{12} - 1) > R_{13}R_{14}$ and $\frac{\varepsilon_1}{\alpha_1} < \frac{\varepsilon_2}{\sigma_2}$
$E_3 = (0, N_{1E}, 0, N_{2E})$	$\frac{\varepsilon_1}{\alpha_1} > \frac{\varepsilon_2}{\sigma_2}, \frac{\varepsilon_2}{\alpha_2} > \frac{\varepsilon_1}{\sigma_1}$ and $\frac{\sigma_1}{\alpha_1} > \frac{\alpha_2}{\sigma_2}$	$A_{11} > 0, A_{31} > 0, A_{41} > 0$ and $A_{11}A_{21}A_{31} > A_{31}^2 + A_{11}^2 A_{41}$
	(or all inequalities are reversed)	
$E_4 = (X_{1E}, N_{1E}, X_{2E}, N_{2E})$		$A_{12} > 0, A_{32} > 0, A_{42} > 0$ and $A_{12}A_{22}A_{32} > A_{32}^2 + A_{12}^2 A_{42}$

Biological Interpretation

1. If there are no species initially, then there are never any species going to extinction.

2. If the growth rate of species 2 and the intraspecific competition in species 1 are relatively small compared to the growth rate of species 1 and the adverse effect species 1 has on species 2 , the basic reproduction number R_{11} and R_{12} are below the threshold and $(R_{11}-1)(R_{12}-1) > R_{13}R_{14}$, then species 2 goes to extinction, the disease dies out and species 1 goes to its carrying capacity $\dfrac{\varepsilon_1}{\sigma_1}$.

3. If the growth rate of species 1 and the intraspecific competition in species 2 are relatively small compared to the growth rate of species 2 and the adverse effect species 2 has on species 1 , the basic reproduction number R_{11} and R_{12} are below the threshold and $(R_{11}-1)(R_{12}-1) > R_{13}R_{14}$, then species 1 goes to extinction, the disease dies out and species 2 goes to its carrying capacity $\dfrac{\varepsilon_2}{\sigma_2}$.

4. If the growth rate of species 2 and the adverse effect species 2 has on species 1 are relatively small compared to the growth rate of species 1 and the intraspecific competition in species 2, the growth rate of species 1 and the adverse effect species 1 has on species 2 are relatively small compared to the growth rate of species 2 and the intraspecific competition in species 1, the adverse effect species 1 has on species 2 and the adverse effect species 2 has on species 1 are relatively small compared to the intraspecific competition in species 1 and the intraspecific competition in species 2 (or all are reversed) and $A_{11} > 0, A_{31} > 0, A_{41} > 0$ and $A_{11}A_{21}A_{31} > A_{31}^2 + A_{11}^2A_{41}$, then the disease dies out and both species go to their usual persistent equilibria.

5. If $A_{12} > 0, A_{32} > 0, A_{42} > 0$ and $A_{12}A_{22}A_{32} > A_{32}^2 + A_{12}^2A_{42}$, then species 1 and 2 go to their usual persistent equilibria and the disease persists in both species.

Competition *SIS* Model with Mass Action Incidence

Equilibrium	Existence?	Stable?
$E_0 = (0,0,0,0)$	Yes	No
$E_1 = \left(0, \frac{\varepsilon_1}{\sigma_1}, 0, 0\right)$	Yes	$R_{21} < 1$ and $\frac{\varepsilon_2}{\alpha_2} < \frac{\varepsilon_1}{\sigma_1}$
$E_2 = \left(\frac{\varepsilon_1}{\sigma_1}\left(1 - \frac{1}{R_{21}}\right), \frac{\varepsilon_1}{\sigma_1}, 0, 0\right)$	$R_{21} > 1$	$\frac{\varepsilon_2}{\alpha_2} < \frac{\varepsilon_1}{\sigma_1}$
$E_3 = \left(0,0,0,\frac{\varepsilon_2}{\sigma_2}\right)$	Yes	$R_{22} < 1$ and $\frac{\varepsilon_1}{\alpha_1} < \frac{\varepsilon_2}{\sigma_2}$
$E_4 = \left(0,0,\frac{\varepsilon_2}{\sigma_2}\left(1 - \frac{1}{R_{22}}\right), \frac{\varepsilon_2}{\sigma_2}\right)$	$R_{22} > 1$	$\frac{\varepsilon_1}{\alpha_1} < \frac{\varepsilon_2}{\sigma_2}$
$E_5 = (0, N_{1E}, 0, N_{2E})$	$\frac{\varepsilon_1}{\alpha_1} > \frac{\varepsilon_2}{\sigma_2}, \frac{\varepsilon_2}{\alpha_2} > \frac{\varepsilon_1}{\sigma_1}$ and $\frac{\sigma_1}{\alpha_1} > \frac{\alpha_2}{\sigma_2}$	$A_{13} > 0, A_{33} > 0, A_{43} > 0$ and $A_{13}A_{23}A_{33} > A_{33}^2 + A_{13}^2 A_{43}$
	(or all inequalities are reversed)	
$E_6 = (I_{1E}, N_{1E}, I_{2E}, N_{2E})$		$A_{14} > 0, A_{34} > 0, A_{44} > 0$ and $A_{14}A_{24}A_{34} > A_{34}^2 + A_{14}^2 A_{44}$

Biological Interpretation

1. If there are no species initially, then there are never any species going to extinction.

2. If the growth rate of species 2 and the intraspecific competition in species 1 are relatively small compared to the growth rate of species 1 and the adverse effect species 1 has on species 2 and the basic reproduction number R_{21} is below the threshold, then species 2 goes to extinction, the disease dies out and species 1 goes to its carrying capacity $\frac{\varepsilon_1}{\sigma_1}$.

3. If the growth rate of species 2 and the intraspecific competition in species 1 are relatively small compared to the growth rate of species 1 and the adverse effect species 1 has on species 2 and the basic reproduction number R_{21} is above the threshold, then species 2 goes to extinction, the disease in species 1 approaches the endemic level and species 1 goes to its carrying capacity $\frac{\varepsilon_1}{\sigma_1}$.

4. If the growth rate of species 1 and the intraspecific competition in species 2 are relatively small compared to the growth rate of species 2 and the adverse effect species 2 has on species 1 and the basic reproduction number R_{22} is below the threshold, then species 1 goes to extinction, the disease dies out and species 2 goes to its carrying capacity $\frac{\varepsilon_2}{\sigma_2}$.

5. If the growth rate of species 1 and the intraspecific competition in species 2 are relatively small compared to the growth rate of species 2 and the adverse effect species 2 has on species 1 and the basic reproduction number R_{22} is above the threshold, then species 1 goes to extinction, the disease in species 2 approaches the endemic level and species 2 goes to its carrying capacity $\frac{\varepsilon_2}{\sigma_2}$.

6. If the growth rate of species 2 and the adverse effect species 2 has on species 1 are relatively small compared to the growth rate of species 1 and the intraspecific competition in species 2, the growth rate of species 1 and the adverse effect species 1 has on species 2 are relatively small compared to the growth rate of species 2 and the intraspecific competition in species 1, the adverse effect species 1 has on species 2 and the adverse effect species 2 has on species 1 are relatively small compared to the intraspecific competition in species 1 and the intraspecific competition in species 2 (or all are reversed) and $A_{13} > 0, A_{33} > 0, A_{43} > 0$ and $A_{13}A_{23}A_{33} > A_{33}^2 + A_{13}^2 A_{43}$, then the disease dies out and both species go to their usual persistent equilibria.

7. If $A_{14} > 0, A_{34} > 0, A_{44} > 0$ and $A_{14}A_{24}A_{34} > A_{34}^2 + A_{14}^2 A_{44}$, then species 1 and 2 go to their usual persistent equilibria and the disease persists in both species.

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