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Classification of Spherically Symmetric Non Static Space-Times According to their Killing Vector Fields

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Abstract

In this research work, we classify Spherically Symmetric Non Static Space-times according to their isometries. To do this, we use direct integration techniques. Using the above techniques, we find three different cases in which the above quoted symmetry exists. The dimension of isometries in first two cases is three while in third case it is six.

Keywords: Isometry, lie algebra and direct integration technique

1. Introduction

Symmetry made many complicated things easy in general relativity. Space-times are classified according to different symmetries in general relativity while each classification gives different information about the space-times due to which one can easily study each of their aspects. We are looking here for killing symmetry in spherically symmetric non static space times by using direct integration techniques. We label here a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature $(-,+,+,+)$ by M. The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by R^{a}_{bcd}, the Ricci tensor components are $R_{ab} = R^{c}_{acb}$ and the Weyl tensor components are C^{a}_{bcd}. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skewsymmetrization, respectively. The decomposition of the covariant derivative of any vector field X on M is as follows

$$X_{a:b} = \frac{1}{2}h_{ab} + F_{ab} \qquad (1.1)$$

Where $h_{ab}(= h_{ba})$ and $F_{ab}(= -F_{ba})$ are symmetric and skew symmetric tensors on M respectively. A Killing vector field X on a manifold M is the solution of killing equation $\underset{X}{L}\, g_{ab} = 0$ which can also be written as [3, 4, 6, 7]

$$g_{ab,c}X^c + g_{ac,b}X^c + g_{bc,a}X^c = 0 \qquad (1.2)$$

We study here the Lie algebra of killing vector fields admitted by the metric. The structure constant C^a_{bc} in terms of Lie brackets is given by [1, 2, 5]

$$[X_b, X_c] = C^a_{bc}\, X_a, \quad C^a_{bc} = -C^a_{cb}, \qquad (1.3)$$

2. Main Results

We consider here the spherically symmetric non static space-times in usual coordinates (t, r, θ, ϕ) (labeled by (x^0, x^1, x^2, x^3), respectively) with line element [2] $\quad ds^2 = -e^{A(t,r)}dt^2 + e^{B(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (2.1)$

For the above isometry we write equation (1.2) explicitly to get the following ten equations

$$A_t(t,r)X^0 + A_r(t,r)X^1 + 2X^0_{,0} = 0 \qquad (2.2)$$

$$e^{A(t,r)}X^0_{,1} - e^{B(t,r)}X^1_{,0} = 0 \qquad (2.3)$$

$$e^{A(t,r)}X^0_{,2} - r^2 X^2_{,0} = 0 \qquad (2.4)$$

$$e^{A(t,r)}X^0_{,3} - r^2\sin^2\theta\, X^3_{,0} = 0 \qquad (2.5)$$

$$B_t(t,r)X^0 + B_r(t,r)X^1 + 2X^1_{,1} = 0 \qquad (2.6)$$

$$e^{B(t,r)}X^1_{,2} + r^2 X^2_{,1} = 0 \qquad (2.7)$$

$$e^{B(t,r)}X^1_{,3} + r^2\sin^2\theta\, X^3_{,1} = 0 \qquad (2.8)$$

$$X^1 + rX^2_{,2} = 0 \qquad (2.9)$$

$$X^2_{,3} + \sin^2\theta\, X^3_{,2} = 0 \qquad (2.10)$$

$$\sin\theta\, X^1 + r\cos\theta\, X^2 + r\sin\theta\, X^3_{,3} = 0 \qquad (2.11)$$

Dividing (2.11) by $r\sin\theta$ and subtract from (2.9), we get the following equation

$$X^2_{,2} - \cot\theta\, X^2 - X^3_{,3} = 0 \qquad (2.12)$$

One can find the following system if one differentiates equations (2.4), (2.5) and (2.10) with respect to ϕ and θ by using equation (2.9)

$$X^0 = r^2 e^{-A(t,r)} \sin\theta \int E_t^1(t,r,\phi) d\phi + E^3(t,r,\theta),$$
$$X^1 = r\sin\theta \int E^1(t,r,\phi) d\phi - rE_\theta^4(r,\theta,\phi) + E^5(t,r,\theta),$$
$$X^2 = \cos\theta \int E^1(t,r,\phi) d\phi + \frac{1}{r^2} \int e^{A(t,r)} E_\theta^3(t,r,\theta) dt + E^4(r,\theta,\phi), \quad (2.13)$$
$$X^3 = \cos ec\theta E^1(t,r,\phi) + E^2(r,\theta,\phi),$$

where $E^1(t,r,\phi)$, $E^2(r,\theta,\phi)$, $E^3(t,r,\theta)$, $E^4(r,\theta,\phi)$ and $E^5(t,r,\theta)$ are functions of integration which are to be determined.
Using equations (2.11), (2.12) and (2.9) in system (2.13) we have the information $E^1(t,r,\phi) = F^1(t,r)\cos\phi + F^2(t,r)\sin\phi + F_\phi^3(r,\phi)$, $E^3(t,r,\theta) = re^{-A(t,r)}\cos\theta F_t^6(t,r) + F^5(t,r)$ and $E^5(t,r,\theta) = [F^6(t,r) - \frac{Q^1(r)}{r}]\cos\theta.$ Plugging back these information in system (2.13), we have the following refine system

$$X^0 = r^2 e^{-A(t,r)} \sin\theta [F_t^1(t,r)\sin\phi - F_t^2(t,r)\cos\phi] + re^{-A(t,r)}\cos\theta F_t^6(t,r) + F^5(t,r)$$
$$X^1 = r\sin\theta[F^1(t,r)\sin\phi - F^2(t,r)\cos\phi] + r\sin\theta F^3(r,\phi) - rE_\theta^4(r,\theta,\phi) + $$
$$+ [F^6(t,r) - \frac{Q^1(r)}{r}]\cos\theta, \quad (2.14)$$
$$X^2 = [F^1(t,r)\sin\phi - F^2(t,r)\cos\phi + F^3(r,\phi)]\cos\theta - [\frac{1}{r}F^6(t,r) + \frac{1}{r^2}Q^1(r)]\sin\theta$$
$$+ E^4(r,\theta,\phi)$$
$$X^3 = \cos ec\theta[F^1(t,r)\cos\phi + F^2(t,r)\sin\phi] + \cos ec\theta F_\phi^3(r,\phi) + E^2(r,\theta,\phi).$$

Differentiate (2.8) with respect to t and $\theta,$ using the above system (2.14) we get

$$B_t(t,r)[\cos ec\theta E_{\theta\phi}^4(r,\theta,\phi)]_{,\theta} = 0 \quad (2.15)$$

Equation (2.15) gives the following three possibilities

(I) $B_t(t,r) = 0 \, and \, [\cos ec\theta E_{\theta\phi}^4(r,\theta,\phi)]_{,\theta} \neq 0.$

(II) $B_t(t,r) \neq 0 \, and \, [\cos ec\theta E_{\theta\phi}^4(r,\theta,\phi)]_{,\theta} = 0.$

(III) $B_t(t,r) = 0 \, and \, [\cos ec\theta E_{\theta\phi}^4(r,\theta,\phi)]_{,\theta} = 0.$

Case (I) gives contradiction after a lengthy calculation if one can proceed further.
Case (II): Here we have
$B_t(t,r) = 0 \, and \, [\cos ec\theta E_{\theta\phi}^4(r,\theta,\phi)]_{,\theta} \neq 0 \Rightarrow B(t,r) = B(r)$. Using the remaining six equations in system (2.14), we get the following solution

$$X^0 = 0, \; X^1 = 0, \; X^2 = c_1\cos\phi + c_2\sin\phi,$$
$$X^3 = -c_1\sin\phi\cot\theta + c_2\cos\phi\cot\theta + c_5, \quad (2.16)$$

where $c_1, c_2, c_3, c_4, c_5 \in R.$ The above space-times in this case take the form

$$ds^2 = -e^{A(t,r)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (2.17)$$

The space-time (2.17) admits three linearly independent killing vector fields having the following generators

$$X_1 = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi},\ X_2 = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}, X_3 = \frac{\partial}{\partial\phi}. \quad (2.18)$$

The above generators satisfying the following Lie algebra

$$[X_1, X_2] = -X_3,\ [X_1, X_3] = X_2,\ [X_2, X_3] = -X_1,\ [X_i, X_j] = 0 \text{ (otherwise)}. \quad (2.19)$$

Case (III): In this case $B_t(t,r) = 0$ *and* $[\cos ec\theta\, E^4_{\theta\phi}(r,\theta,\phi)]_{,\theta} = 0 \Rightarrow B(t,r) = B(r)$ and $E^4(r,\theta,\phi) = -\cos\theta F^1(r,\phi) + F^2(r,\theta) + F^3(r,\phi)$. Plugging back the above information and using equations (2.8), (2.10), (2.11), (2.6) and (2.16) in system (2.14), the given case is further divided in to three sub cases. One of these sub cases gives contradiction while the other two give the results. To avoid the lengthy calculation, we mention the solutions only.

Case (III(a)):

$$X^0 = e^{-\frac{1}{2}f(t)},\ X^1 = 0,\ X^2 = c_4\cos\phi + c_5\sin\phi,$$
$$X^3 = -c_4\sin\phi\cot\theta + c_5\cos\phi\cot\theta + c_6, \quad (2.20)$$

with metric of the space-time, generators of the killing symmetry and corresponding Lie algebra identical to that of the metric (2.17), subject to the constraints $A_{tr}(t,r) = 0 \Rightarrow \quad A(t,r) = f(t) + g(r),\ rB_r(r) + 2 - 2e^{B(r)} \neq 0$ where $f(t)$ and $g(r)$ are any functions of t and r respectively. Here $c_4, c_5, c_6 \in R.$

Case (III(b)):

$$X^0 = e^{-\frac{A(t)}{2}},$$
$$X^1 = r\sin\theta\, e^{-\int\frac{e^{B(r)}}{r}dr}(c_7\sin\phi - c_8\cos\phi) - rc_9\cos\theta\, e^{-\int\frac{e^{B(r)}}{r}dr},$$
$$X^2 = \cos\theta\, e^{-\int\frac{e^{B(r)}}{r}dr}(c_7\sin\phi - c_8\cos\phi) + c_9\sin\theta\, e^{-\int\frac{e^{B(r)}}{r}dr}$$
$$+ c_{10}\cos\phi + c_{11}\sin\phi, \quad (2.21)$$
$$X^3 = \cos ec\theta\, e^{-\int\frac{e^{B(r)}}{r}dr}(c_7\cos\phi + c_8\sin\phi) - c_{10}\cot\theta\sin\phi$$
$$+ c_{11}\cot\theta\cos\phi + c_{12},$$

with constraints $r\cos\theta\, e^{-\int\frac{e^{B(r)}}{r}dr}(c_{25}\sin\phi - c_{26}\cos\phi) + rc_{30}\sin\theta\, e^{-\int\frac{e^{B(r)}}{r}dr} \neq 0,$ $rB_r(r) + 2 - 2e^{B(r)} = 0,\ A_r(t,r) = 0 \Rightarrow A = A(t)$ and $c_7, c_8, c_9, c_{10}, c_{11}, c_{12} \in R.$

The space-times (2.1) become as

$$ds^2 = -e^{A(t)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (2.22)$$

The above space-time (2.22) admits six linearly independent killing vector fields whose generators are given in the following equation

$$X_1 = e^{-\int \frac{e^{B(r)}}{r}dr}(r\sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{\partial}{\partial\theta} + \cos ec\theta\cos\phi\frac{\partial}{\partial\phi}),$$

$$X_2 = -e^{-\int \frac{e^{B(r)}}{r}dr}(r\sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{\partial}{\partial\theta} - \cos ec\theta\sin\phi\frac{\partial}{\partial\phi}), \tag{2.23}$$

$$X_3 = e^{-\int \frac{e^{B(r)}}{r}dr}(-r\cos\theta\frac{\partial}{\partial r} + \sin\theta\frac{\partial}{\partial\theta}),\ X_4 = (\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}),$$

$$X_5 = (\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}),\ X_6 = \frac{\partial}{\partial\phi}.$$

Following are the Lie algebra satisfied by the above generators

$$[X_1, X_2] = (1-e^{B(r)})e^{-2\int e^{\frac{B(r)}{r}dr}} X_6, [X_1, X_3] = (1-e^{B(r)})e^{-2\int e^{\frac{B(r)}{r}dr}} X_5,$$

$$[X_1, X_5] = X_3, [X_1, X_6] = [X_3, X_4] = X_2,\ [X_2, X_3] = -(1-e^{B(r)})e^{-2\int e^{\frac{B(r)}{r}dr}} X_4, \tag{2.24}$$

$$[X_2, X_4] = -X_3, [X_2, X_6] = [X_3, X_5] = -X_1, [X_4, X_5] = -X_6,$$

$$[X_4, X_6] = X_5, [X_5, X_6] = -X_4, [X_i, X_j] = 0\ \text{(otherwise)}.$$

3. Conclusion

In this research work, we study killing symmetry of spherically symmetric non static space-times by using direct integration techniques and find three possible cases. Case (I) gives a contradiction while the remaining two cases give the said symmetry. In case (II) the dimension of the killing symmetry is three while the space-time, generators and corresponding Lie algebra for this case are given respectively in equations (2.17), (2.18) and (2.19). There are two types of solutions in case (III) which are given in case (III(a)) and case (III(b)). The dimension; space-time and Lie algebra in the first sub case of case (III) are identical to that of case (II) while the second sub case of case (III) gives a killing symmetry of dimension six. The space-time, generators and Lie algebra in this second sub case of case (III) are mentioned in equations (2.22), (2.23) and (2.24) respectively.

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