Applied Mathematical Sciences, Vol. 6, 2012, no. 53, 2625 – 2647

Maximum Principle and Error Estimate of the Upwind Finite Volumes Method for Nonlinear Convection-Diffusion Equation

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Abstract. In this paper we used the combined F.E.-F.V. method to solve an initial-boundary value problem, provided that the nonlinear convective term is approximated by U.F.V. method on the finite volume mesh dual to a triangular grid of weakly acute type and the diffusion term is approximated by standard Galerkin finite element method. We proved the stability and the discrete maximum principle under the condition $0 \le \tau \le \left|\hat{\Omega}_i\right| / c(M)\left|\partial\hat{\Omega}_i\right|,\ i \in J$. Also proved the approximate solution is convergent with error of $0(\tau)$ and the local truncation with error of $0(\tau^2)$.

Keywords: finite volumes, Maximum Principle, Error estimate, Convection-Diffusion problem

1. Introduction

Convection-diffusion processes appear in many areas of science and technology, and environmental protection. These problems are peculiar in the sense that it is a combination of two dissimilar phenomena, convection and diffusion. It is a well-known fact that the use of classical Galerkin method with continuous piecewise linear finite elements leads to spurious oscillations when the local Peclet number is large. This is the reason that the numerical solution of convection-diffusion problems attracts a number of specialists. From an extensive literature devoted to problems let us mention the papers **[1], [6]**, **[12]**, **[11]**, **[10], [**3**],** and **[5]**. In the theory of weak solutions for partial differential equations in divergence from there are two roughly equivalent formulations in common use, namely, the functional formulation involving integration against smooth test functions versus the finite volume type over arbitrary control volumes. The former corresponds to energy methods and leads naturally to F.E discretizations for elliptic and parabolic, i.e., diffusive, problems. The latter corresponds in a natural way to the physical formulation of the basic laws of mass, momentum, and energy in fluid mechanics leading directly to the well-known **FV** methods. In **[9]** the work reported here we investigate the approach by trying to have the best of both worlds, i.e., the combination of finite volumes for in viscid conservation laws with finite elements for the diffusion. Computational fluid dynamics problems usually require discretization of the problem into a large number of cells/grid points (millions and more), therefore cost of the solution favors simpler, lower order approximation within each cell/grid point. This is especially true of " external flow " problems, like air flow around the car or airplane, or weather simulation in a large area. We view space as being broken down into a set of volumes each of which surrounds one point. In particular the volume associated with each point is the region of space that is closer to that point than to any other point. Now we consider how can be used these cells to solve fluid flow problem. The way to think about it is to consider the total flow that enters a cell through its boundaries. It is claimed that the net flow (sum of flow in and out) has to be zero. Intuitively this is because the fluid (or air) has nowhere to go. This condition can be written formally as

$$\int_{\partial\Omega_i} \vec{u}.\vec{n}ds = 0$$

where $\partial\Omega_i$ is the boundary of the i^{th} cell, $\vec{u}$ is the flow and $\vec{n}$ is the vector normal to the surface let us mention the papers **[11], [10], and [9]**. Our main goal is to develop a robust theoretically based numerical method for the solution of viscous compressible flow applied of unstructured meshes. In this paper we proposed numerical schemes for the solution of viscous gas flow based on the combination of the **FV** method for the discretization of inviscid convective terms and the **FE** method applied to the approximation of viscous terms.
This paper consists

2. The Convection-Diffusion Equation

We consider the two–dimensional nonlinear convection–diffusion equation **[6]**.

$$\frac{\partial u}{\partial t} - \lambda \Delta u + \nabla.(\vec{b}(u).u) = f(x,t) \qquad \Omega \times (0,T) = Q_T\ , \tag{2.1}$$

$$u(x,t) = 0, \qquad \text{on } \partial\Omega\ , \tag{2.2}$$

$$u(x,0) = u^o(x), \qquad \text{on } \bar{\Omega} \tag{2.3}$$

where $x = (x_1, x_2)$ and $\lambda > 0$ is a given constant, the vector $\vec{b}(u) : Q_T \to \mathrm{R}^2$ is a convection coefficient, the functions $f(x,t) : Q_T \to \mathrm{R}$ ($f = f(x,t)$ for simplicity) and the function $u^o : \Omega \to \mathrm{R}$ are given. The weak form is to find $u : [0,T] \to H_0^1(\Omega)$, for simplicity we will denote $u = u(t) = u(.,t)$:

$$(\frac{d}{dt} u(t), v) + (\lambda \nabla u(t), \nabla v) + (\nabla.(\vec{b}(u(t)).u(t)), v) = (f, v),\ \forall v \in H_0^1(\Omega)\ ,$$

$$u(0) = u^0. \tag{2.4}$$

Assumption 2.1. [11]

a) $f \in C(\ [0,T], W^{1,q}(\Omega)\ )$ for some $q > 2$.

b) $u^0 \in W_0^{1,p}(\Omega)$ for some $p > 2$.

c) a positive constants γ_1 and γ_2 such that

$$\left\| u^0 \right\|_{L^\infty(\Omega)} \le \gamma_1\ , \quad \left\| f \right\|_{L^\infty(Q_T)} \le \gamma_2\ .$$

Now, we define the set $Y = \{u : |u| \le M\}$ and the coefficients of equation (2.1) satisfy the following conditions

A (2.1) $\vec{b}(u) = (b_1(u), b_2(u)) \ \in \ W^{1,\infty}(Y) \times W^{1,\infty}(Y)$.

A (2.2) $\vec{b}(u)$ is locally Lipschitz –continuous $\left|\vec{b}(u) - \vec{b}(v)\right| \le L|u - v|, \quad \forall u, v \in Y$

Where $0 < L < 1$ is Lipchitz constant?

A (2.3) $u \in L^\infty(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega))$, $u' \in L^\infty(0,T; L^2(\Omega))$ and $u'' \in L^\infty(0,T; V^*)$,

where u is the weak solution of equation (2.4) and V^* is the dual space of V.

Lemma 2.1. [5] Let U^k and u^k are the approximate solution and the exact solution Respectively, if $\left\| U^k - u^k \right\|_{L^2(\Omega)} \le Ch$, $t_k \in (0,T)$ then

$$\left\| \nabla U^k \right\|_{L^2(\Omega)} \le C_1,$$

where C_1 is a constant independent of h and τ.

Lemma 2.2. [8] Let $\varphi \in H^2(\Omega) \cap V$ and $\varphi_h = p_h \varphi$ such that \

$$\|p_h\varphi - \varphi\|_{H_0^1(\Omega)} \leq ch\,\|\varphi\|_{H^2(\Omega)}, \qquad h \in (0, h_o)$$

where p_h is the projection and the constant c is independent of φ and h.

Lemma 2.3. [10] Let $\varphi \in H^2(\Omega) \cap V$ and $\varphi_h = p_h \varphi$ such that

$$\|p_h\varphi - \varphi\|_{L^2(\Omega)} \leq ch^2\|\varphi\|_{H^2(\Omega)} \qquad h \in (0, h_o),$$

where p_h is the projection and the constant c is independent of φ and h.

3. Finite Elements Triangulations [11]

The F.E. method, in its simplest form, is a specific process of constructing subspaces $V_h \subset V$, which are called F.E. spaces. The most characteristic is discretization such as triangulation $\mathcal{T}_h$ established over the closure $\bar{\Omega} = \Omega \cup \partial\Omega$ of $\Omega \subset \mathbb{R}^2$. Let Ω_h be a polygonal approximation of the domain Ω, thus denotes a partition of Ω into disjoint triangles e such that the following properties are satisfied

i) $\bar{\Omega}_h = \bigcup_{e \in \mathcal{T}_h} e,$

ii) If $e_1, e_2 \in \mathcal{T}_h$, then $e_1^o \cap e_2^o = \phi$ or e_1 and e_2 have a common side or e_1 and e_2 have a common vertex,

iii) $P \in \bar{\Omega}_h$ for any vertex P of each $e \in \mathcal{T}_h$.

Then the triangulation $\mathcal{T}_h$ is called a basic mesh.

Remark 3.1[11]. We associate the index set $J = \{i \neq j; p_i \in \bar{\Omega}_h\}$ and $\overset{o}{J} = \{i \in J; p_i \in \Omega_h\}$ be the set of the indices of all interior vertices. Let $\mathcal{P}_h = \{p_i; i \in J\}$ be the set of all vertices of all $e \in \mathcal{T}_h$. Let ϕ_i, $i \in J$ be the continuous function in $\bar{\Omega}_h$ such that $\phi_i(p)$ is a linear on each $e \in \mathcal{T}_h$ and $\phi_i(p_j) = \delta_{ij}$ for all p_j, $i, j \in J$, where δ_{ij} is Kronecker`s delta. The linear F.E. spaces V_h and V_{0h}

$$V_h = \{\phi : \phi \in C(\bar{\Omega}_h), \phi \text{ is linear function for each } e \in \mathcal{T}_h\}.$$
$$V_{0h} = \{\phi : \phi \in V_h, \phi = 0 \text{ on } \partial\Omega_h\}.$$

By I_h we denote the operator of the interpolation $I_h : C(\bar{\Omega}_h) \to V_{0h}$. Hence, if $v : \mathcal{P}_h \to \mathbb{R}$, then $I_h v \in V_{0h}, \quad (I_h v)(p_i) = v(p_i), \quad p_i \in \mathcal{P}_h$.

4. The Dual Finite Volumes

To construct the dual mesh $\hat{\Omega}_h = \left\{ \hat{\Omega}_i ; i \in J \right\}$ over the basic mesh $\mathcal{T}_h$, the dual F.V. $\hat{\Omega}_i$ associated with a vertex $P_i \in \mathcal{P}_h$ is a closed polygon obtained in the following way **[11]** : By joining the center of every triangle $e \in \mathcal{T}_h$ that contains the vertex P_i with the center of every side of e containing P_i. If $P_i \in \mathcal{P}_h \cap \partial\Omega_h$, then complete the obtained contour by the straight segments joining P_i with the centers of boundary sides (i.e. sides which are subsets of $\partial\Omega_h$) that contain P_i. In this way, one can get the boundary $\partial\hat{\Omega}_i$ of the finite volume $\hat{\Omega}_i$ (see Figure 4.1). It is obvious that

$$\bar{\Omega}_h = \bigcup_{i \in J} \hat{\Omega}_i .$$

The interiors of $\hat{\Omega}_i, i \in J$, are mutually disjoint. If for two different F.V. $\hat{\Omega}_i$ and $\hat{\Omega}_j$ their boundaries contain a common straight segment, which call neighbors, then

$$\Gamma_{ij} = \bigcup_{\alpha=1}^{\ell_{ij}} \Gamma_{ij}^{\alpha} = \partial\hat{\Omega}_i \cap \partial\hat{\Omega}_j = \Gamma_{ji.},$$

where the set Γ_{ij} consists of one or two straight segments Γ_{ij}^{α} and $\Gamma_{ij}^{\alpha} = \Gamma_{ji}^{\alpha}$ as shown in Figure **4.1**. We have

$$\ell_{ij} = \begin{cases} 2 & \text{for } \hat{\Omega}_i \text{ or } \hat{\Omega}_j \subset \hat{\Omega}_h \\ 1 & \text{for } \hat{\Omega}_i \text{ and } \hat{\Omega}_j \text{ are adjacent to } \partial\hat{\Omega}_h. \end{cases}$$

If $P_i \in \mathcal{P}_h \cap \partial\hat{\Omega}_h$, then we denote by $\Gamma_{i,-1}^{\alpha} = \ell_{i,-1}$ for $\alpha = 1,2$, the segments that form $\partial\hat{\Omega}_i \cap \partial\hat{\Omega}_h$. Put

$$S(i) = \begin{cases} s(i) \cup \{-1\} & \text{for } P_i \in \mathcal{P}_h \cap \partial\hat{\Omega}_h \\ s(i) & \text{for } P_i \in \mathcal{P}_h \cap \hat{\Omega}_h \end{cases} .$$

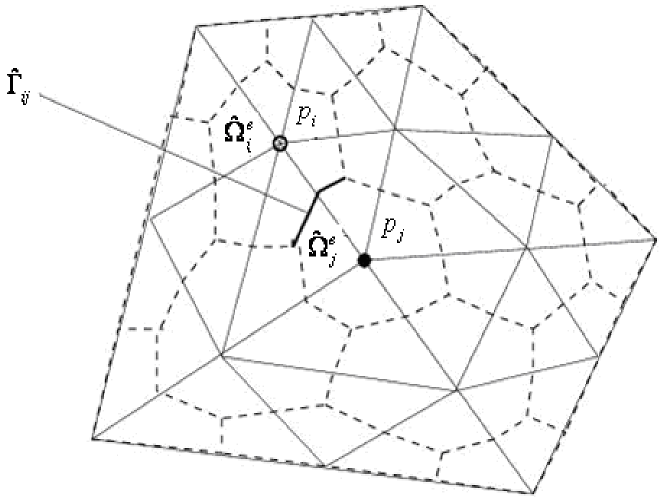

Figure 4.1: Basic triangular mesh in a domain $\hat{\Omega}_h$ and the dual F.V.

For every $\hat{\Omega}_i \in \hat{\Omega}_h$ we have

$$\partial\hat{\Omega}_i = \bigcup_{j\in S(i)} \Gamma_{ij} = \bigcup_{j\in S(i)} \bigcup_{\alpha=1}^{\ell_{ij}} \Gamma_{ij}^{\alpha}. \tag{4.1}$$

the open segments obtained by removing the endpoints from Γ_{ij}^{α} are mutually disjoint.

Remark 4.1. We introduce the following notations

a) $|e| =$ area of $e \in \mathcal{T}_h$.

b) $\beta_{ij} = |\Gamma_{ij}| =$ length of Γ_{ij}, $\beta_{ij}^{\alpha} = |\Gamma_{ij}^{\alpha}|$.

c) $|\hat{\Omega}_i| =$ area of $\hat{\Omega}_i \in \hat{\Omega}_h$, $|\partial\hat{\Omega}_i| =$ length of $\partial\hat{\Omega}_i$.

d) $n_{ij}^{\alpha} = (n_{1ij}^{\alpha}, n_{2ij}^{\alpha}) =$ unit outer normal to $\partial\hat{\Omega}_i$ on Γ_{ij}^{α}.

e) Let us consider a partition $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = T$ of the time interval $(0,T)$ and set $\tau = t_{k+1} - t_k$ for $k = 0,1,\ldots$.

Assumptions 4.1. [11]

a) Triangulation family $\mathcal{T}_h$ is of weakly acute type. This means all angles of all $e \in \mathcal{T}_h$, are less than or equal to $\pi/2$.

b) Assume that the domain Ω is polygonal and thus $\hat{\Omega}_h = \Omega$.

c) Define $h_e =$ the length of the longest side of the triangle $e \in \mathcal{T}_h$,

θ_e =the magnitude of the smallest angle of the triangle $e \in \mathcal{T}_h$,

and put $h = \max_{e \in \mathcal{T}_h} h_e$, $\theta_h = \min_{e \in \mathcal{T}_h} \theta_e$.

d) Triangulation family $\mathcal{T}_h$, $h \in (0, h_0)$, be regular, i.e. there exists $\vartheta_0 > 0$ such that, $\theta_h \ge \vartheta_0 > 0 \qquad \forall h \in (0, h_0)$.

e) In view of Assumption **(c)** which implies the existence of a constant $\hat{c} > 0$ such that

$$h^2 \le \hat{c}|e|, \qquad e \in \mathcal{T}_h,\ h \in (0, h_0).$$

f) Introduce the inverse stability assumption $h = 0(\tau)$ it means that we consider the condition $\quad h \le \breve{c}\,\tau$

where a constant $\breve{c} > 0$ independent of h and τ.

5. The Upwind Finite Volumes Scheme

Let $\tau > 0$ be the time step, the U.F.V. scheme for $t_k \in [0, T)$ to find $U^{k+1} \in V_{0h}$ such that

$$(\frac{U^{k+1} - U^k}{\tau}, v_h)_h + (\lambda \nabla U^{k+1}, \nabla v_h)_h + R(\vec{B}^k, U^k, v_h) = (f^{k+1}, v_h)_h, \quad \forall v_h \in V_{0h},$$
$$U^0 = I_h u^0, \tag{5.1}$$

Where I_h is the interpolation operator and $R(\vec{B}^k, U^k, v)$ is derived as follows: By using (4.1) and the definition of the mass lamping operator L_h for all $u, v \in V_h$ we get **[13]**,:

$$(\nabla.(\vec{b}(u).u), v) = \int_{\Omega} \nabla.(\vec{b}(u).u) v dx \approx \sum_{i \in J} \int_{\hat{\Omega}_h} \nabla.(\vec{b}(u).u) L_h v dx$$
$$= \sum_{i \in J} v(p_i) \int_{\hat{\Omega}_i} \nabla.(\vec{b}(u).u) dx = \sum_{i \in J} v(p_i) \int_{\partial \hat{\Omega}_i} \vec{b}(u) nuds$$
$$= \sum_{i \in J} v(p_i) \sum_{j \in S(i)} \sum_{\alpha=1}^{\ell_{ij}} \int_{\Gamma_{ij}^{\alpha}} \vec{b}(u) nuds, \tag{5.2}$$

replacing the function u on Γ_{ij}^{α} by some convex combination of the nodal values of u_i and u_j with parameter H_{ij}

$$(\nabla.(\vec{b}(u).u), v) \approx \sum_{i \in J} v(p_i) \sum_{j \in s(i)} \sum_{\alpha=1}^{\ell_{ij}} (H_{ij} u_i + H_{ji} u_j) \int_{\Gamma_{ij}^{\alpha}} \vec{b}(u) n_{ij}^{\alpha} ds,$$

then we define

$$R(\vec{B}^k, U^k, v) = \sum_{i\in J} v(p_i) \sum_{j\in s(i)} \sum_{\alpha=1}^{\ell_{ij}} (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \beta_{ij}^{k,\alpha}, \tag{5.3}$$

where $U_i^k = U^k(p_i)$, $v_i = v(p_i)$ and

$$\beta_{ij}^{k,\alpha} = \int_{\Gamma_{ij}^\alpha} \vec{B}^k n_{ij}^\alpha ds, \quad \vec{B}^k = \vec{b}(U^k).$$

Where the function $H: \mathbb{R}^2 \times S \longrightarrow \mathbb{R}$, $S = \{n \in \mathbb{R}^2, |n| = 1\}$ is called a numerical flux.

With $a_{ij}^k = (\lambda \nabla \phi_j, \nabla \phi_i), \quad i, j \in J$

where ϕ_i, ϕ_j are the basis functions in $H_0^1(\Omega)$.

5.1.The Properties of the Numerical Flux ([11] , [7]) We use the following assumptions

a) $H = H(y, z, n)$ is locally Lipschitz-continuous with respect to y, z for any $M \geq 0$ there exists a constant $c(M) > 0$ such that

$$|H(y,z,n) - H(y^*, z^*, n)| \leq c(M)(|y - y^*| + |z - z^*|) \quad \forall y, y^*, z, z^* \in [-M, M].$$

b) H is consistent

$$H(u,u,n) = \vec{b}(u)nu, \quad \forall u \in V.$$

c) H is conservative

$$H(y,z,n) = -H(z,y,-n) \qquad \forall y, z \in V.$$

d) H is monotone in the following sense: For a given fixed number $M > 0$, the function $H(y,z,n)$ is non-increasing with respect to the second variable z on the set $\zeta_M = \{(y,z,n); y,z \in [-M, M]\}$.

5.2. The Basic Properties of the Discrete Problem [11].

a) If $i \in J$ and $e \in \mathcal{T}_h$ is a triangle with the vertex $p_i \in \mathcal{P}_h$ then

$$|e \cap \hat{\Omega}_i| = \frac{1}{3}|e|.$$

b) The approximation $(.,.)_h$ of the L^2-scalar product can be defined with the aid of numerical integration using the vertices P_1^e, P_2^e, P_3^e of $e \in \mathcal{T}_h$, as the integration points

$$(u,v)_h = \sum_{e\in\mathcal{T}_h} |e| \sum_{k=1}^{3} u(P_k^e)\, v(P_k^e)/3 = \int_\Omega (L_h u)(L_h v) dx, \quad u, v \in C(\bar{\Omega}_h),$$

$$\|u\|_{L^2(\Omega)} = \|L_h u\|_{L^2(\Omega)}, \quad u \in C(\bar{\Omega}_h).$$

c) We have

$$(\phi_j,\phi_i)_h = \begin{cases} \left|\hat{\Omega}_i\right|, & i = j \\ 0\ , & i \neq j \end{cases}$$

$$(u,\phi_i)_h = \frac{1}{3}\sum_{\{e\in\mathcal{T}_h;\, p_i\in e\cap\mathcal{P}_h\}} |e|\, u(p_i) = \left|\hat{\Omega}_i\right| u(p_i), \qquad i \in J,\ u \in H_0^1(\Omega),$$

$$(f^k,\phi_i)_h = \left|\hat{\Omega}_i\right| f(p_i,t_k), \qquad i \in J,\ t_k \in [0,T].$$

Remark 5.1.

a) from assumption (4.1-**b**) we find $(\nabla.,\nabla.) = (\nabla.,\nabla.)_h$.

b) By c we denote a generic constant independent of $h,\tau,k,...$ which attains in general different values in different places.

6. The Maximum Principle [14]

The DMP plays an important role in computational partial differential equations by guaranteeing that approximation of physically nonnegative quantities.

Lemma 6.1. If $\tau > 0$ time step and $h \in (0,h_0)$ then

$$0 \le \tau \le \left|\hat{\Omega}_i\right| \Big/ c(M)\left|\partial\hat{\Omega}_i\right|, \qquad i \in J, c(M) > o. \tag{6.1}$$

Proof. From the property (5.2 –**a**)

$$\left|e \cap \hat{\Omega}_i\right| = \frac{1}{3}|e| = \left|\hat{\Omega}_i\right| \Rightarrow \frac{1}{3}|e| \Big/ \left|\partial\hat{\Omega}_i\right| = \left|\hat{\Omega}_i\right| \Big/ \left|\partial\hat{\Omega}_i\right|, \tag{6.2}$$

from the assumption (4.1-**e**) we find

$$|e| \ge \frac{h^2}{\hat{c}} \quad \Rightarrow \quad \frac{1}{3}|e| \Big/ \left|\partial\hat{\Omega}_i\right| = h^2 \Big/ 3\hat{c}\left|\partial\hat{\Omega}_i\right| \quad e \in \mathcal{T}_h,\ h \in (0,h_0),$$

then equation (6.2) becomes

$$\left|\hat{\Omega}_i\right| \Big/ \left|\partial\hat{\Omega}_i\right| \ge h^2 \Big/ 3\hat{c}\left|\partial\hat{\Omega}_i\right|,$$

since $\left|\partial\hat{\Omega}_i\right| \ge h$, we get

$$\left|\hat{\Omega}_i\right| \Big/ \left|\partial\hat{\Omega}_i\right| \ge h/3\hat{c} \ge \ = \hat{c}_1 h \qquad \text{where } \hat{c}_1 = 1/3\hat{c}, \tag{6.3}$$

we will consider the stability condition(see **[4]**)

$$0 < \tau \le \hat{c}_1 c(M)^{-1} h, \tag{6.4}$$

then from equations (6.3) and (6.4) , we obtain (6.1). □

Lemma 6.2 [13]. let $A=(a_{ij}),\ C=(c_{ij}),\ D=(d_{ij})$ be $N\times K$ matrices satisfying the conditions

1. $\sum_{j\in J} a_{ij} = \sum_{j\in J} c_{ij} = \sum_{j\in J} d_{ij} > 0$. for $i \in \overset{o}{J}$,

2. $c_{ij} \geq 0$, for $i\in \overset{o}{J},\ j\in J$,

3. $d_{ij} \geq 0$, for $i\in \overset{o}{J},\ j\in J$,

4. $a_{ij} \leq 0$, for $i\in \overset{o}{J},\ j\in J,\ j\neq i$,

and assume that $A\vec{u} = C\vec{w} + \tau D\vec{g}$,

then each component $u_i (i\in \overset{o}{J})$ is estimated by $\max_{j\in J}|u_j| \leq \max_{j\in J}|w_j| + \tau \max_{j\in J}|g_j|$.

Theorem 6.1. If $\tau > 0$ and $h\in(0,h_0)$ satisfy the condition (6.1) and if assumption (2.1-**c**) holds then the solution of equation (5.1) is bounded and estimated by

$$\|U^k\|_{L^\infty(\Omega)} \leq \|u^0\|_{L^\infty(\Omega)} + T\|f\|_{L^\infty(Q_T)} \leq M .$$

Proof. We prove the theorem by mathematical induction. Clearly the theorem is valid for $k=0$

$$\|U^0\|_{L^\infty(\Omega)} = \|I_h u^0\|_{L^\infty(\Omega)} \leq \|u^0\|_{L^\infty(\Omega)} \leq M,$$

we assume that the theorem is valid for m $(0\leq m\leq k)$,

$$\|U^m\|_{L^\infty(\Omega)} \leq \|u^0\|_{L^\infty(\Omega)} + m\tau\|f\|_{L^\infty(Q_T)} \leq M.$$

The matrix form of (5.1)given by

let $L_h U(x,t) = \sum_{j\in J} U(p_i)\phi_j$ and choose $v_h = \phi_i(x)$, $i\in \overset{o}{J}$, one can get

$$U^{k+1}(p_i)\sum_{j\in J}(\phi_j,\phi_i)_h - U^k(p_i)\sum_{j\in J}(\phi_j,\phi_i)_h + \tau U^{k+1}(p_i)\sum_{j\in J}(\lambda\nabla\phi_j,\nabla\phi_i)_h$$

$$= -\tau U^k(p_i)\sum_{j\in J} R(\vec{B}^k,\phi_j,\phi_i) + \tau f(p_i,t_{k+1})\sum_{j\in J}(\phi_j,\phi_i)_h\ ,\ i\in \overset{o}{J}\ ,$$

let $W=(w_{ij}),\ S=(s_{ij})$ and $B=(b_{ij}^k)$, are the matrices of mass, Stiffness and convection, respectively which have the components:

$$w_{ij} = (\phi_j,\phi_i)_h = \frac{1}{3}\sum_{\{e\in\mathcal{T}_h;\, p_i\in e\cap\mathcal{P}_h]} |e| = |\hat{\Omega}_i|,\ s_{ij} = (\lambda\nabla\phi_j,\nabla\phi_i)_h,\ b_{ij}^n = -R(\vec{B}^k,\phi_j,\phi_i)\ .$$

Then the matrix form of equation (5.1)

$$[W+\tau\ S]U^{k+1}(p_i)=[W+\tau\ B]U^{k}(p_i)+\tau\ Wf(p_i,t_{k+1}),\ \ i\in J.$$

We have **[13]**.

$$w_{ii}>0\quad w_{ij}=0\ (i\neq j);\qquad \sum_{j\in J}s_{ij}=0,\ \ s_{ij}\leq 0\ (i\neq j),$$

it is easy to see

$$(w_{ij}+\tau\ s_{ij})\leq 0\,,\quad (i\in \overset{o}{J},\ \ j\in J,\ \ i\neq j)\ \text{and}\ \ \sum_{j\in J}(w_{ij}+\tau\ s_{ij})>0\,,\ (i\in \overset{o}{J}),$$

$$b_{ij}^{k}=-R(\vec{B}^{k},\phi_j,\phi_i)$$

$$=\sum_{j\in s(i)}\sum_{\alpha=1}^{\ell_{ij}}\left[(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_i^{k})-(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_j^{k})-(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_i^{k})\right]\beta_{ij}^{k,\alpha}$$

in view of the property (5.1-**b**) of H and $H_{ij}^{k}+H_{ji}^{k}=1$ (see **[13]**),

$$\sum_{j\in s(i)}\sum_{\alpha=1}^{\ell_{ij}}(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_i^{k})\beta_{ij}^{k,\alpha}\ =\sum_{j\in s(i)}\sum_{\alpha=1}^{\ell_{ij}}(H_{ij}^{k}U_i^{k}+U_i^{k}-H_{ij}^{k}U_i^{k})\beta_{ij}^{k,\alpha}$$

$$=\sum_{j\in s(i)}\sum_{\alpha=1}^{\ell_{ij}}U_i^{k}\beta_{ij}^{k,\alpha}\ =\int_{\partial\Omega_i}\vec{B}^{k}U_i^{k}n\,ds=0\,,$$

then

$$b_{ij}^{k}=\sum_{j\in s(i)}\sum_{\alpha=1}^{\ell_{ij}}\left[(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_i^{k})-(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_j^{k})\right]\beta_{ij}^{k,\alpha}\ ,$$

hence, if **[11]**

$$\mathcal{H}_{ij}=\begin{cases}0 & U_i^{k}=U_j^{k}\\ \displaystyle\sum_{\alpha=1}^{\ell_{ij}}\frac{(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_i^{k})-(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_j^{k})}{U_j^{k}-U_i^{k}}\beta_{ij}^{k,\alpha} & U_i^{k}\neq U_j^{k}\end{cases}\ ,$$

then in case $U_i^{k}\neq U_j^{k}$ we get

$$\sum_{\alpha=1}^{\ell_{ij}}[(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_i^{k})-(H_{ij}^{k}U_i^{k}+H_{ji}^{k}U_j^{k})]\,\beta_{ij}^{k,\alpha}=\mathcal{H}_{ij}(U_j^{k}-U_i^{k})\,,$$

can be written as $\quad b_{ij}^{k}=\sum_{j\in s(i)}\mathcal{H}_{ij}(U_j^{k}-U_i^{k})\,,$

by the property (5.1–**d**) of H, $\mathcal{H}_{ij}\geq 0$, $i\in \overset{o}{J}$, $j\in s(i)$ then $\quad b_{ij}^{k}\geq 0$.

From the proof of **[5,** theorem **3.5.1]**, gives $\quad \sum_{j\in J}b_{ij}^{k}=0\,,\quad i\in \overset{o}{J}$,

now, we want to prove $(w_{ij}+\tau b_{ij}^{k})\geq 0$, by the property (5.1-**a**) of H

$$\left|(H_{ij}^k U_i^k + H_{ji}^k U_i^k) - (H_{ij}^k U_i^k + H_{ji}^k U_j^k)\right| \le c(M)(\left|U_i^k - U_i^k\right| + \left|U_i^k - U_j^k\right|)$$
$$= c(M)\left|U_i^k - U_j^k\right|,$$

then $$\mathcal{H}_{ij} \le \sum_{\alpha=1}^{\ell_{ij}} \frac{c(M)\left|U_i^k - U_j^k\right|}{U_j^k - U_i^k}\beta_{ij}^{k,\alpha} \le c(M)\sum_{\alpha=1}^{\ell_{ij}}\beta_{ij}^{k,\alpha},$$

so, $$0 \le \mathcal{H}_{ij} \le c(M)\sum_{\alpha=1}^{\ell_{ij}}\beta_{ij}^{k,\alpha} = c(M)\left|\Gamma_{ij}\right|,$$

by using (4.1) we have, $0 \le \sum_{j\in s(i)} \mathcal{H}_{ij} \le c(M)\left|\partial\hat{\Omega}_i\right|,$

hence, the condition (6.1), gives

$$0 \le \sum_{j\in s(i)} \mathcal{H}_{ij} \le c(M)\left|\partial\hat{\Omega}_i\right| \le \frac{\left|\hat{\Omega}_i\right|}{\tau}, \quad i \in \overset{o}{J} \quad \Rightarrow \quad \left|\hat{\Omega}_i\right| - \tau\sum_{j\in s(i)} \mathcal{H}_{ij} \ge 0, \qquad i \in \overset{o}{J},$$

it is easy to see that

$$(w_{ij} + \tau b_{ij}^k) = ([\left|\hat{\Omega}_i\right| - \tau\sum_{j\in s(i)} \mathcal{H}_{ij}]U_i^k + \tau\sum_{j\in s(i)} \mathcal{H}_{ij}\, U_j^k) \ge 0, \ (i \in \overset{o}{J}, j \in J, i \ne j),$$

then by lemma **6.2** we get,

$$\max_{j\in J}\left|U_j^{k+1}\right| \le \max_{j\in J}\left|U_j^k\right| + \tau\max_{j\in J}\left|f_j^{k+1}\right|,$$

Since $\left\|U^0\right\|_{L^\infty(\Omega)} = \left\|I_h u^0\right\|_{L^\infty(\Omega)} \le \left\|u^0\right\|_{L^\infty(\Omega)} \le \gamma_1$ and $\|f\|_{L^\infty(Q_T)} \le \gamma_2$, by induction assumption we get

$$\left\|U^{k+1}\right\|_{L^\infty(\Omega)} \le \left\|U^0\right\|_{L^\infty(\Omega)} + (k+1)\tau\|f\|_{L^\infty(Q_T)}. \qquad \square$$

Lemma 6.3. If c^* is a positive constant and $c(M) > 0$ such that

$$\left|R(\vec{B}^k, U^k, v)\right| \le c^*|v|_{H_0^1(\Omega)}, \qquad u \in V_h \cap L^\infty(\Omega),\ v \in V_h,\ h \in (0, h_0)$$

Proof. By using Theorem **6.1**, the property (5.1-**c**) of H, the relations ($\Gamma_{ij}^\alpha = \Gamma_{ji}^\alpha$, $\beta_{ij}^{k,\alpha} = \beta_{ji}^{k,\alpha}$ and $n_{ij}^\alpha = -n_{ji}^\alpha$), (5.3) and use the property (5.1-**b**) of H, can be get

$$R(\vec{B}^k, U^k, v) = \left| \frac{1}{2} \sum_{i \in J} \sum_{j \in s(i)} \sum_{\alpha=1}^{\ell_{ij}} (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \beta_{ij}^{k,\alpha} \right| [v(p_i) - v(p_j)]$$

$$= \left| \frac{1}{2} \sum_{i \in J} \sum_{j \in s(i)} \sum_{\alpha=1}^{\ell_{ij}} (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \beta_{ij}^{k,\alpha} - \frac{1}{2} \sum_{i \in J} \sum_{j \in s(i)} \sum_{\alpha=1}^{\ell_{ij}} (H_{ij}^k U^k(0) + H_{ji}^k U^k(0)) \beta_{ij}^{k,\alpha} \right| [v(p_i) - v(p_j)]$$

then by using the property 5.1-**a** of H we find

$$\left| R(\vec{B}^k, U^k, v) \right| \leq c(M) \sum_{i \in J} \sum_{j \in s(i)} \sum_{\alpha=1}^{\ell_{ij}} \left| U^k \right| \beta_{ij}^{k,\alpha} [v(p_i) - v(p_j)],$$

since $\beta_{ij}^{k,\alpha} \leq \frac{h}{3}$ and $\left| v(p_i) - v(p_j) \right| \leq h \left| \nabla v \right|_{e_{ij}^{\alpha}} \Big|$ then

$$\left| R(\vec{B}^k, U^k, v) \right| \leq \frac{c(M)}{3} \max_{\bar{\Omega}} \left| U^k \right| \sum_{i \in J} \sum_{j \in s(i)} \sum_{\alpha=1}^{\ell_{ij}} h^2 \left| \nabla v \right|_{e_{ij}^{\alpha}} \Big|, \tag{6.5}$$

from Assumption (4.1-**e**) and the fact that each $e \in \mathcal{T}_h$ appears in the above sum as some e_{ij}^{α} at most six times, theorem **6.1** and from Cauchy-Schwartz inequality we conclude that

$$\left| R(\vec{B}^k, U^k, v) \right| \leq 2\hat{c}c(M) \left\| U^k \right\|_{L^\infty(\Omega)} \int_\Omega \left| \nabla v \right| dx = 2\hat{c}c(M)(\text{meas}(\Omega))^{1/2} \left\| U^k \right\|_{L^\infty(\Omega)} \left| v \right|_{H_0^1(\Omega)},$$

with $c^* = 2\hat{c}c(M)M(\text{meas}(\Omega))^{1/2}$. □

7. The Stability of Approximate Solution

Theorem 7.1. There exists a constant $\delta > 0$ independent of h, τ and m such that

$$\left\| U^m \right\|_{L^2(\Omega)} \leq \delta$$

Proof. In view of theorem **6.1**, Assumption 2.1-**c** and the condition (6.4). If we set $v_h = U^{k+1}$ in (5.1) then

$$(U^{k+1} - U^k, U^{k+1})_h + \tau(\lambda \nabla U^{k+1}, \nabla U^{k+1})_h + \tau R(\vec{B}^k, U^k, U^{k+1}) = \tau(f^{k+1}, U^{k+1})_h, \ \tau > 0 \tag{7.1}$$

we will assume the relation

$$(y - z, y)_h = \frac{1}{2}\left(\left\| y \right\|_{L^2(\Omega)}^2 - \left\| z \right\|_{L^2(\Omega)}^2 + \left\| y - z \right\|_{L^2(\Omega)}^2 \right),$$

valid for $y, z \in C(\bar{\Omega})$, we get

$$(U^{k+1} - U^k, U^{k+1})_h = \frac{1}{2}\left(\|U^{k+1}\|^2_{L^2(\Omega)} - \|U^k\|^2_{L^2(\Omega)} + \|U^{k+1} - U^k\|^2_{L^2(\Omega)} \right),$$

then (7.1) becomes

$$\|U^{k+1}\|^2_{L^2(\Omega)} - \|U^k\|^2_{L^2(\Omega)} + \|U^{k+1} - U^k\|^2_{L^2(\Omega)} + 2\tau\lambda |U^{k+1}|^2_{H^1_0(\Omega)}$$
$$= 2\tau(f^{k+1}, U^{k+1})_h - 2\tau R(\vec{B}^k, U^k, U^{k+1})_h, \; t_k \in [0, T) \tag{7.2}$$

in virtue of Lemma **6.3** and Young's inequality we have

$$2\left|R(\vec{B}^k, U^k, U^{k+1})\right| \le \frac{(c^*)^2}{\varepsilon} + \varepsilon |U^{k+1}|^2_{H^1_0(\Omega)}, \tag{7.3}$$

by the property (5.2-**d**) and Cauchy-Schwartz inequality we find

$$\left|(f^{k+1}, U^{k+1})_h\right| = \left|(L_h f^{k+1}, L_h U^{k+1})_h\right| \le \|L_h f^{k+1}\|_{L^2(\Omega)} \|L_h U^{k+1}\|_{L^2(\Omega)}, \tag{7.4}$$

now, by using the definition of the mass lamping operator L_h in **[13]** we get

$$\|L_h f^{k+1}\|_{L^2(\Omega)} = \left(\int_\Omega \left(\sum_{i \in J} f^{k+1}(p_i) \hat{\mu}_i(p) \right)^2 \right)^{1/2} \le c \|f^{k+1}\|_{C([0,T]; W^{1,q}(\Omega))} \tag{7.5}$$

further, we use **[13**, lemma **2.2]** and from the definition **1.2** in **[2]** we get

$$\|L_h U^{k+1}\|_{L^2(\Omega)} \le \tilde{c}_2 c_1^{-1} |U^{k+1}|_{H^1_0(\Omega)}, \tag{7.6}$$

from (7.5), (7.6) and Young's inequality, we have

$$2\left|(f^{k+1}, U^{k+1})\right| \le \tilde{c}_2^2 c_1^{-2} \|f^{k+1}\|^2_{C([0,T]; W^{1,q}(\Omega))} / \varepsilon + \varepsilon |U^{k+1}|^2_{H^1_0(\Omega)}, \tag{7.7}$$

now, by substituting (7.3), (7.7) in (7.2), choosing $\varepsilon = \frac{\lambda}{2}$ and using Remark (5.1-**b**), we get

$$\|U^{k+1}\|^2_{L^2(\Omega)} - \|U^k\|^2_{L^2(\Omega)} + \|U^{k+1} - U^k\|^2_{L^2(\Omega)} + \tau\lambda |U^{k+1}|^2_{H^1_0(\Omega)} \le c\tau \quad t_k \in [0, T),$$

where $c = 2\left(\tilde{c}_2^2 c_1^{-2} \|f^{k+1}\|^2_{C([0,T]; W^{1,q}(\Omega))} + (c^*)^2 / \lambda \right)$. Summation over $k = 0, \ldots, m-1$, $t_m \in (0, T]$ then

$$\|U^m\|^2_{L^2(\Omega)} + \sum_{k=0}^{m-1} \|U^{k+1} - U^k\|^2_{L^2(\Omega)} + \tau\lambda \sum_{k=0}^{m-1} |U^{k+1}|^2_{H^1_0(\Omega)} \le cT + \|U^0\|^2_{L^2(\Omega)}, \tag{7.8}$$

since $U^0 = I_h u^0$ and by using the definition **1.2** in **[2]** then

$$\left\|U^0\right\|^2_{L^2(\Omega)} \le \tilde{c}_2^2 c_1^{-2}\left|I_h u^0\right|^2_{H_0^1(\Omega)},$$

then (7.8) becomes

$$\left\|U^m\right\|^2_{L^2(\Omega)} + \sum_{k=0}^{m-1}\left\|U^{k+1}-U^k\right\|^2_{L^2(\Omega)} + \tau\lambda\sum_{k=0}^{m-1}\left|U^{k+1}\right|^2_{H_0^1(\Omega)} \le cT + \tilde{c}_2^2 c_1^{-2}\left|I_h u^0\right|^2_{H_0^1(\Omega)} \le \delta, \quad t_m \in (0,T]$$

the last inequality is a consequence of the bounded

$$\left|I_h u^0\right|^2_{H_0^1(\Omega)} \le c, \qquad h \in (0,h_0)$$

and the second and third terms are nonnegative, , we obtain $\left\|U^m\right\|_{L^2(\Omega)} \le \delta$. □

8. The Local Truncation Error

We suppose that the exact solution $u:(0,T)\to V$ of (2.4) satisfies the condition **A(3.3)**. Hence $u \in C([0,T];L^2(\Omega)) \cap C^1([0,T];V^*) \cap L^\infty(Q_T)$ and set $M = \| u^k\|_{L^\infty(Q_T)} < \infty$.

Lemma 8.1. Under conditions **A(3.3)**, for $t_n \in [0,T)$ we have

(a) $\left|(u^{k+1}-u^k,v) - \tau(u'(t_{k+1}),v)\right| \le c\tau^2\|v\|_{H_0^1(\Omega)}, \quad v \in V,$

(b) $\left|(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v) - (\nabla.(\vec{b}(u^k).u^k),v)\right| \le c\tau|v|_{H_0^1(\Omega)}, \; v \in V.$

Proof. **(a)** We based on the following result **[10]**:
If $\omega:(0,T)\to V^*$ is such that $\omega,\omega' \in L^1(0,T;V^*)$ and $v \in V$ then $\langle\omega',v\rangle \in L^1(0,T)$ and $\int_{t_1}^{t_2} \langle\omega'(t),v\rangle dt = \langle\omega(t_2)-\omega(t_1),v\rangle, \quad t_1,t_2 \in [0,T]$. Here $\langle\varphi,v\rangle$ denotes the value of a functional $\varphi \in V^*$ at a point $v \in V$. A similar result holds, if $\omega,\omega' \in L^1(0,T;L^2(\Omega))$ and the duality $\langle .,.\rangle$ is replaced by the L^2-scalar product $(.,.)$. Let $v \in V$, then

$$\begin{aligned}(u^{k+1}-u^k,v) - \tau(u'(t_{k+1}),v) &= (u(t_{k+1})-u(t_k),v) - \int_{t_k}^{t_{k+1}} \left(u'(t_{k+1}),v\right)dt. \\ &= \int_{t_k}^{t_{k+1}} \langle u'(t)-u'(t_{k+1}),v\rangle dt \end{aligned} \tag{8.1}$$

Since $u'' \in L^\infty(0.T;V^*)$

$$\langle u'(t)-u'(t_{k+1}),v\rangle = \int_{t_{k+1}}^{t} \langle u''(\omega),v\rangle d\omega, \tag{8.2}$$

by substituting (8.2) in (8.1) and using Cauchy-Schwartz inequality, we get

$$\left|(u^{k+1}-u^k,v)-\tau(u'(t_{k+1}),v)\right| = \left|\int_{t_k}^{t_{k+1}}\left(\int_{t_{k+1}}^{t}\langle u''(\omega),v\rangle d\omega\right)dt\right|$$

$$= \left|\tau\int_{t_k}^{t_{k+1}}\langle\frac{u'(t)-u'(t_{k+1})}{\tau},v\rangle dt\right| = \left|\tau\int_{t_k}^{t_{k+1}}\langle u''(\omega),v\rangle dt\right|$$

$$= \left|\tau^2(\frac{u'(t_{k+1})-u'(t_k)}{\tau},v)\right| = \tau^2\left|(u''(t),v)\right| \le c\tau^2\|v\|_{H_0^1(\Omega)}.$$

(b) By using Green's theorem then

$$\left|(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v)-(\nabla.(\vec{b}(u^k).u^k),v)\right| = \left|(\vec{b}(u^{k+1}).u^{k+1},\nabla v)-(\vec{b}(u^k).u^k,\nabla v)\right|$$

adding and subtracting $(\vec{b}(u^{k+1}).u^k,\nabla v)$, by using Cauchy-Schwartz inequality, Theorem **8.1** and the conditions **A(3.1)–A(3.3)**, we get

$$\left|(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v)-(\nabla.(\vec{b}(u^k).u^k),v)\right| \le \left|(\vec{b}(u^{k+1}).(u^{k+1}-u^k),\nabla v)\right|+\left|((\vec{b}(u^{k+1})-\vec{b}(u^k)).u^k,\nabla v)\right|$$

$$\le \xi\|u^{k+1}-u^k\|_{L^2(\Omega)}|v|_{H_0^1(\Omega)}, \tag{8.3}$$

since $u'\in L^\infty(0,T;L^2(\Omega))$

$$\|u^{k+1}-u^k\|_{L^2(\Omega)} \le \tau\|u'\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \le c\tau,$$

then (8.3) becomes

$$\left|(\nabla.(\vec{b}(u^{k+1})u^{k+1}),v)-(\nabla.(\vec{b}(u^k)u^k),v)\right| \le c\tau|v|_{H_0^1(\Omega)}, \qquad v\in V.$$

Theorem 8.2. If u^k and U^k are solutions of (2.4) and (5.1), respectively at time $t=t_k$ and u^k satisfies the conditions A(3.3) then

$$\text{L.T.E} \le 0(\tau^2), \qquad \tau>0.$$

Proof. The exact solution u satisfies (2.4) at $t=t_{k+1}$ then

$$(u'(t_{k+1}),v)+(\lambda\nabla u^{k+1},\nabla v)+(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v)=(f^{k+1},v). \quad \forall v\in V,$$

adding and subtracting $(\frac{u^{k+1}-u^k}{\tau},v)$ and $(\nabla.(\vec{b}(u^k).u^k),v)$ with setting $v=v_h\in V_h$, gives

$$(u^{k+1}-u^k,v_h)+\tau(\lambda\nabla u^{k+1},\nabla v_h)+\tau(\nabla.(\vec{b}(u^k).u^k),v_h)=\tau(f^{k+1},v_h)+$$

$$[(u^{k+1}-u^k,v_h)-\tau(u'(t_{k+1}),v_h)]+\tau[(\nabla.(\vec{b}(u^k).u^k),v_h)-(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v_h)] \; v_h\in V_h, \tag{8.4}$$

by subtracting equation (8.4) from the equation (5.1) and using Remark (5.1-**a**) such that

$$(U^{k+1}-U^{k},v_h)_h-(u^{k+1}-u^{k},v_h)+\tau[(\lambda\nabla U^{k+1},\nabla v_h)-(\lambda\nabla u^{k+1},\nabla v_h)]$$

$$+\tau[R(\vec{B}^{k},U^{k},v_h)-(\nabla.(\vec{b}(u^{k}).u^{k}),v_h)]=\tau[(f^{k+1},v_h)_h-(f^{k+1},v_h)]$$

$$-[(u^{k+1}-u^{k},v_h)-\tau(u'(t_{k+1}),v_h)]-\tau[(\nabla.(\vec{b}(u^{k}).u^{k},v_h)-(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v_h)],\quad \forall v_h\in V_h$$

adding and subtracting $(U^{k+1}-U^{k},v_h)$ and $\tau(\nabla.(\vec{b}(U^{k}).U^{k}),v_h)$, gives

$$(U^{k+1}-u^{k+1},v_h)-(U^{k}-u^{k},v_h)+\tau\lambda(\nabla(U^{k+1}-u^{k+1}),\nabla v_h)$$
$$+\tau[(\nabla.(\vec{b}(U^{k}).U^{k}),v_h)-(\nabla.(\vec{b}(u^{k}).u^{k}),v_h)]=I_1+I_2,\ \forall v_h\in V_h \tag{8.5}$$

where

$$I_1=-[(u^{k+1}-u^{k},v_h)-\tau(u'(t_{k+1}),v_h)]$$
$$-\tau[(\nabla.(\vec{b}(u^{k}).u^{k},v_h)-(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v_h)],$$

$$I_2=\tau[(f^{k+1},v_h)_h-(f^{k+1},v_h)]+[(U^{k+1},v_h)-(U^{k+1},v_h)_h]$$
$$-[(U^{k},v_h)-(U^{k},v_h)_h]+\tau[(\nabla.(\vec{b}(U^{k}).U^{k}),v_h)-R(\vec{B}^{k},U^{k},v_h)].$$

To estimate I_1, we use Lemma **8.1**,

$$|I_1|\ \le\ |(u^{k+1}-u^{k},v_h)-\tau(u'(t_{k+1}),v_h)|+\tau|(\nabla.(\vec{b}(u^{k}).u^{k},v_h)-(\nabla.(\vec{b}(u^{k+1}).u^{k+1}),v_h)|$$
$$\le\ c\tau^2\|v_h\|_{H_0^1(\Omega)}\ \le\ 0(\tau^2),\qquad v_h\in V_h,\ \ \tau>0. \tag{8.6}$$

We can write $I_2=\sum_{i=1}^{4}I_{2i}$. The estimate I_{21} is a consequence of **[12**, Theorem **4.1.5** and the proof of Theorem **4.1.6]**

$$|I_{21}|=\tau|(f^{k+1},v_h)_h-(f^{k+1},v_h)|\ \le \tau ch\|v_h\|_{H_0^1(\Omega)},\quad \forall v_h\in V_h. \tag{8.7}$$

The estimate of I_{22} and I_{23} are a consequence of **[12**, Theorem **3.1.6]** and using Lemma **2.1,** we have

$$|I_{22}|=|(U^{k+1},v_h)-(U^{k+1},v_h)_h|\ \ \le ch^2\|v_h\|_{H_0^1(\Omega)}, \tag{8.8}$$

so we find

$$|I_{23}|\ =|(U^{k},v_h)-(U^{k},v_h)_h|\ \le\ ch^2\|v_h\|_{H_0^1(\Omega)}. \tag{8.9}$$

To estimate $|I_{24}|$, by adding and subtracting $\tau(\nabla.(\vec{b}(U^{k}).U^{k}),L_hv_h)$ then

$$|I_{24}|\ =\ \tau|(\nabla.(\vec{b}(U^{k}).U^{k}),v_h)-R(\vec{B}^{k},U^{k},v_h)|$$

$$\leq \tau\left|(\nabla.(\vec{b}(U^k).U^k),v_h)-(\nabla.(\vec{b}(U^k).U^k),L_hv_h)\right|+\tau\left|(\nabla.(\vec{b}(U^k).U^k),L_hv_h)-R(\vec{B}^k,U^k,v_h)\right|$$

$$= \left|I_{24}^1\right| + \left|I_{24}^2\right|$$

by Cauchy-Schwartz inequality, Lemma **2.1**, Theorem **8.1**, **[13**, Lemma **2.1]** and **A(3.1)** then

$$\left|I_{24}^1\right| \leq \tau\left|(\nabla.(\vec{b}(U^k).U^k),v_h)-(\nabla.(\vec{b}(U^k).U^k),L_hv_h)\right| \leq \tau\left|(\nabla.(\vec{b}(U^k).U^k),v_h-L_hv_h)\right|$$

$$\leq c\tau\, h\, \|v_h\|_{H_0^1(\Omega)}. \tag{8.10}$$

And $\left|I_{24}^2\right| = \tau\left|(\nabla.(\vec{b}(U^k).U^k),L_hv_h)-R(\vec{B}^k,U^k,v_h)\right|$

$$=\tau\frac{1}{2}\sum_{i\in J}\sum_{j\in s(i)}\sum_{\alpha=1}^{\ell_{ij}}\left|\int_{\Gamma_{ij}^\alpha}\vec{b}(U^k)U^k n_{ij}^\alpha ds-(H_{ij}^kU_i^k+H_{ji}^kU_j^k)\beta_{ij}^{k,\alpha}\right|\left|v_h(p_i)-v_h(p_j)\right|,$$

If $i\in J$ and $j\in s(i)$,then the segment p_ip_j a side of triangles $e_{ij}^\alpha\in\mathcal{T}_h$ such that $\Gamma_{ij}^\alpha\subset e_{ij}^\alpha$ then.

$\left|p_i-p_j\right|\leq h$, $\quad \left|x-p_i\right|\leq h$ for $x\in\Gamma_{ij}^\alpha$ $,\beta_{ij}^{k,\alpha}\leq\dfrac{h}{3}$, $\left|u^k(p_i)-u^k(p_j)\right| \leq h\left|\nabla u^k\big|_{e_{ij}^\alpha}\right|$,

$$\left|u^k(x)-u^k(p_i)\right|\leq h\left|\nabla u^k\big|_{e_{ij}^\alpha}\right| \text{ for } x\in\Gamma_{ij}^\alpha,\ \left|v_h(p_i)-v_h(p_j)\right|\leq h\left|\nabla v_h\big|_{e_{ij}^\alpha}\right|, \tag{8.11}$$

by the properties (5.1-**a,b**) of H and (8.11), we obtain

$$\left|\int_{\Gamma_{ij}^\alpha}\vec{b}(U^k)U^k n_{ij}^\alpha ds-(H_{ij}^kU_i^k+H_{ji}^kU_j^k)\beta_{ij}^{k,\alpha}\right|\leq\left|\int_{\Gamma_{ij}^\alpha}\vec{b}(U^k(p_i))U^k(p_i)n_{ij}^\alpha ds-(H_{ij}^kU_i^k+H_{ji}^kU_j^k)\beta_{ij}^{k,\alpha}\right|$$

$$\leq 2c(M)\max_{x\in\Gamma_{ij}^\alpha}\left|U^k(x)-U_i^k\right|\beta_{ij}^{k,\alpha}+c(M)\left|U_i^k-U_j^k\right|\beta_{ij}^{k,\alpha} \leq h^2c(M)\left|\nabla U^k\big|_{e_{ij}^\alpha}\right|,$$

then we have

$$\tau\left|(\nabla.(\vec{b}(u^k).U^k),L_hv_h)-R(\vec{B}^k,U^k,v_h)\right|\leq\frac{1}{2}\sum_{i\in J}\sum_{j\in s(i)}\sum_{\alpha=1}^{\ell_{ij}}h^3c(M)\tau\left|\nabla U^k\big|_{e_{ij}^\alpha}\right|\left|\nabla v_h\big|_{e_{ij}^\alpha}\right|, \tag{8.12}$$

taking into account that each triangle $e\in\mathcal{T}_h$ appears in the sum in (8.12) as some e_{ij}^α at most six times, by using Assumption (4.1-**e**), Cauchy-Schwartz inequality and Lemma **2.1** we conclude that

$$\tau\left|(\nabla.(\vec{b}(U^k).U^k),L_hv_h)-R(\vec{B}^k,U^k,v_h)\right| \leq \hat{c}c(M)\tau\, h\sum_{e\in\mathcal{T}_h}|e|\left|\nabla U^k\right|\left|\nabla v_h\right|\leq c\tau\, h\,\|v_h\|_{H_0^1(\Omega)}, \tag{8.13}$$

now from (8.10) and (8.13) we get

$$\left|I_{24}\right|\leq c\tau\, h\,\|v_h\|_{H_0^1(\Omega)}, \tag{8.14}$$

by using Assumption (4.1-**f**) and from (8.7)- (8.9), (8.14) we get

$$|I_2| \le 0(\tau^2), \tag{8.15}$$

and from (8.6) and (8.15) the proof is complete.

9. The Error Estimate

We define the projection $p_h : V \to V_h$ as follows: If $\varphi \in V$ such that $p_h\varphi \in V_h$,

$$|p_h\varphi|_{H_0^1(\Omega)} \le |\varphi|_{H_0^1(\Omega)}, \quad \varphi \in V. \tag{9.1}$$

Lemma 9.1. For any $M > 0$ there exists a constant $\xi > 0$ such that

$$\left|(\nabla.((\vec{b}(z_1).z_1),v) - (\nabla.(\vec{b}(z_2).z_2),v)\right| \le \xi \|z_1 - z_2\|_{L^2(\Omega)} |v|_{H_0^1(\Omega)},$$

$$z_1, z_2 \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \|z_1\|_{L^\infty(\Omega)}, \|z_2\|_{L^\infty(\Omega)} \le M .$$

Proof. Is similar to proof Lemma 8.1-**b**.

Theorem 9.1. If u^k and U^k are solutions of (2.4) and (5.1) respectively and satisfy the conditions **A(3.3)** and τ be the time step satisfy $0 \le \tau \le |\hat{\Omega}_i| / c(M)|\partial\hat{\Omega}_i|, \quad i \in J$ then

$$\max_{t_k \in [0,T]} \|e^k\|_{L^2(\Omega)} \le 0(\tau), \tag{9.2}$$

where a constant $c > 0$ is independent of h, τ, λ.

Proof. Let $h \in (0, h_o)$ and $\tau > 0$ satisfy the condition (7.4) and Assumption (4.1-**f**), From (8.5), let the error of the method at time $t = t_k$ is defined by

$$e^k = U^k - u^k, \tag{9.3}$$

let $v_h = p_h e^{k+1}$, adding and subtracting (e^{k+1}, e^{k+1}), (e^k, e^{k+1}) and $\tau\lambda(\nabla e^{k+1}, \nabla e^{k+1})$, gives

$$(e^{k+1}, e^{k+1}) - (e^k, e^{k+1}) + \tau\lambda(\nabla e^{k+1}, \nabla e^{k+1}) =$$

$$-\tau[(\nabla.(\vec{b}(U^k).U^k), p_h e^{k+1}) - (\nabla.(\vec{b}(u^k).u^k), p_h e^{k+1})] + I_1 + I_2 + (e^{k+1}, e^{k+1})$$

$$-(e^{k+1}, p_h e^{k+1}) - (e^k, e^{k+1}) + (e^k, p_h e^{k+1}) + \tau\lambda(\nabla e^{k+1}, \nabla e^{k+1}) - \tau\lambda(\nabla e^{k+1}, \nabla p_h e^{k+1})$$

denoting by $I : V \to V$ the identity operator $(I\varphi = \varphi$ for $\varphi \in V)$ then

$$(e^{k+1}-e^k,e^{k+1})+\tau\lambda(\nabla e^{k+1},\nabla e^{k+1})=-\tau[(\nabla.(\vec{b}(U^k).U^k),p_he^{k+1})-(\nabla.(\vec{b}(u^k).u^k),p_he^{k+1})]$$
$$+I_1+I_2+(e^{k+1},(I-p_h)e^{k+1})-(e^k,(I-p_h)e^{k+1})+\tau\lambda(\nabla e^{k+1},\nabla(I-p_h)e^{k+1}), \tag{9.4}$$

from (9.3) it follows that

$$(I-p_h)e^{k+1}=U^{k+1}-u^{k+1}-p_hU^{k+1}+p_hu^{k+1}=p_hu^{k+1}-u^{k+1}, \tag{9.5}$$

hence, by using Lemma **2.2**, Lemma **2.3** and (9.5) then

$$\left\|(I-p_h)e^{k+1}\right\|_{L^2(\Omega)}\le ch^2\left\|u^{k+1}\right\|_{H^2(\Omega)} \tag{9.6}$$

$$\left\|(I-p_h)e^{k+1}\right\|_{H_0^1(\Omega)}\le ch\left\|u^{k+1}\right\|_{H^2(\Omega)} \tag{9.7}$$

by using the relation $A.B=\frac{1}{4}\left[(A+B)^2-(A-B)^2\right]$ and Cauchy-Schwartz inequality, we find

$$(e^{k+1}-e^k,e^{k+1})=\frac{1}{2}\left(\left\|e^{k+1}\right\|^2_{L^2(\Omega)}+\left\|e^{k+1}-e^k\right\|^2_{L^2(\Omega)}-\left\|e^k\right\|^2_{L^2(\Omega)}\right). \tag{9.8}$$

And
$$\left|(\nabla e^{k+1},\nabla e^{k+1})\right|=\left|e^{k+1}\right|^2_{H_0^1(\Omega)}, \tag{9.9}$$

it follows from Lemma **9.1**, (9.1) and (9.3) that

$$\left|(\nabla.(\vec{b}(U^k)U^k),p_he^{k+1})-(\nabla.(\vec{b}(u^k)u^k),p_he^{k+1})\right|\le c\left\|e^k\right\|_{L^2(\Omega)}\left|e^{k+1}\right|_{H_0^1(\Omega)}, \tag{9.10}$$

and since

$$\left|I_1\right|\le c\tau^2\left|e^{k+1}\right|_{H_0^1(\Omega)}, \tag{9.11}$$

$$\left|I_2\right|\le\tau ch\left|e^{k+1}\right|_{H_0^1(\Omega)}+ch^2\left|e^{k+1}\right|_{H_0^1(\Omega)} \tag{9.12}$$

furthermore, by using Cauchy-Schwartz inequality and (9.6), we have

$$\left|(e^{k+1},(I-p_h)e^{k+1})-(e^k,(I-p_h)e^{k+1})\right|\le ch^2\left\|e^{k+1}-e^k\right\|_{L^2(\Omega)}\left\|u^{k+1}\right\|_{H^2(\Omega)}, \tag{9.13}$$

so, by using Cauchy-Schwartz inequality, from the definition **1.2** in **[2]** and (9.6), we have

$$\left|\nabla e^{k+1},\nabla(I-p_h)e^{k+1}\right|\le ch\left|e^{k+1}\right|_{H_0^1(\Omega)}\left\|u^{k+1}\right\|_{H^2(\Omega)}, \tag{9.14}$$

now, by substituting (9.8)–(9.14) in (9.4). Taking into account conditions **A(3.3)** we obtain the estimate

$$\left\|e^{k+1}\right\|^2_{L^2(\Omega)}+\left\|e^{k+1}-e^k\right\|^2_{L^2(\Omega)}-\left\|e^k\right\|^2_{L^2(\Omega)}+2\tau\lambda\left|e^{k+1}\right|^2_{H_0^1(\Omega)}\le c\tau\left\|e^k\right\|_{L^2(\Omega)}\left|e^{k+1}\right|_{H_0^1(\Omega)}$$

$$+c\tau^2\left|e^{k+1}\right|_{H_0^1(\Omega)}+\tau ch\left|e^{k+1}\right|_{H_0^1(\Omega)}+ch^2\left|e^{k+1}\right|_{H_0^1(\Omega)}+2ch^2\left\|e^{k+1}-e^k\right\|_{L^2(\Omega)}+\tau\lambda ch\left|e^{k+1}\right|_{H_0^1(\Omega)},$$

We use (6.4) and assumption (4.1-**f**) then

$$\left\|e^{k+1}\right\|_{L^2(\Omega)}^2+\left\|e^{k+1}-e^k\right\|_{L^2(\Omega)}^2-\left\|e^k\right\|_{L^2(\Omega)}^2+2\tau\lambda\left|e^{k+1}\right|_{H_0^1(\Omega)}^2\leq c\tau\left\|e^k\right\|_{L^2(\Omega)}\left|e^{k+1}\right|_{H_0^1(\Omega)}$$

$$+c\tau h\left|e^{k+1}\right|_{H_0^1(\Omega)}+c\tau h\left|e^{k+1}\right|_{H_0^1(\Omega)}+c\tau h\left|e^{k+1}\right|_{H_0^1(\Omega)}+2ch^2\left\|e^{k+1}-e^k\right\|_{L^2(\Omega)}+\tau\lambda ch\left|e^{k+1}\right|_{H_0^1(\Omega)},$$

by using Young's inequalities we find that

$$\left\|e^{k+1}\right\|_{L^2(\Omega)}^2+\left\|e^{k+1}-e^k\right\|_{L^2(\Omega)}^2-\left\|e^k\right\|_{L^2(\Omega)}^2+2\tau\lambda\left|e^{k+1}\right|_{H_0^1(\Omega)}^2\leq\frac{c\tau}{\lambda}\left\|e^k\right\|_{L^2(\Omega)}^2+\frac{\tau\lambda}{4}\left|e^{k+1}\right|_{H_0^1(\Omega)}^2$$

$$+\frac{\tau\lambda}{4}\left|e^{k+1}\right|_{H_0^1(\Omega)}^2+\frac{c\tau h^2}{\lambda}+\frac{\tau\lambda}{4}\left|e^{k+1}\right|_{H_0^1(\Omega)}^2+ch^4+\left\|e^{k+1}-e^k\right\|_{L^2(\Omega)}^2+\tau\lambda ch^2+\frac{\tau\lambda}{4}\left|e^{k+1}\right|_{H_0^1(\Omega)}^2,$$

and by using Assumption (4.1-**f**) , since the third term is nonnegative we have

$$\left\|e^{k+1}\right\|_{L^2(\Omega)}^2\leq A\left\|e^k\right\|_{L^2(\Omega)}^2+\tau B,\qquad t_k\in[0,T), \tag{9.15}$$

where

$$A=1+\frac{c\tau}{\lambda},\qquad B=c\left[\frac{h^2}{\lambda}(1+\lambda^2)+h^3\right],$$

by induction over $k=0,1,2,...,$ from (9.15) we easily deduce that

$$\left\|e^k\right\|_{L^2(\Omega)}^2\leq A^k\left\|e^0\right\|_{L^2(\Omega)}^2+\tau(A^{k-1}+A^{k-2}+...+1)B,$$

multiplying and dividing the second term by $(A-1)$

$$\left\|e^k\right\|_{L^2(\Omega)}^2\leq A^k\left\|e^0\right\|_{L^2(\Omega)}^2+\tau B\frac{A^k-1}{A-1},\qquad t_k\in[0,T),$$

since $A\leq\exp(\frac{c\tau}{\lambda}),$ then

$$\left\|e^k\right\|_{L^2(\Omega)}^2\leq\exp(\frac{ct_k}{\lambda})\left\|e^0\right\|_{L^2(\Omega)}^2+c\left[\frac{h^2}{\lambda}(1+\lambda^2)+h^3\right]\left(\exp(\frac{ct_k}{\lambda})-1\right)\lambda,$$

taking into account that $u^o\in H_0^1(\Omega)\subset W_0^{1,p}(\Omega),$ by virtue of **[8**, Theorem **2.27]** and the conditions **A(3.3)** we have

$$\left\|e^0\right\|_{L^2(\Omega)}^2\leq ch^2\left\|u^0\right\|_{H^2(\Omega)}^2,$$

also we obtain the estimate

$$\left\|e^k\right\|_{L^2(\Omega)}\leq c\exp(\frac{ct_k}{2\lambda})h+c\left[h(1+\lambda^2)^{1/2}+\sqrt{\lambda}h^{3/2}\right]\left(\exp(\frac{ct_k}{\lambda})-1\right)^{1/2},\ t_k\in[0,T],$$

now $\max_{t_k \in [0,T]} \left\| e^k \right\|_{L^2(\Omega)} \leq \left(c\exp(\frac{cT}{\lambda}) + c(1+\lambda)\exp(\frac{cT}{\lambda}) \right) h + c\sqrt{\lambda} h^{3/2} \exp(\frac{cT}{\lambda})$,

finally, by using Assumption (4.1-**f**) , which immediately implies (9.2). □

Conclusions

In this paper we have analyzed the U.F.V. method for solving nonlinear convection-diffusion equation. This method satisfies the convergence, the discrete maximum principle and the discrete conservation law. We derive the U.F.V. method for the convection term $\nabla.(\vec{b}(u).u)$ which is discretized on the barysentric finite volume mesh dual to a triangular grid of weakly acute type under some assumptions on the regularity of the exact solution of the continuous problem. The discrete maximum principle implying the $L^\infty -$estimate and the stability of approximate solutions are the basic tools used in the investigation of error estimate are proved under the condition $0 \leq \tau \leq \left|\hat{\Omega}_i\right| \big/ c(M)\left|\partial\hat{\Omega}_i\right|,\ i \in J$.

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Received: December, 2011