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RBF-Pseudospectral Method for the Numerical Solution of Good Boussinesq Equation

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Abstract

RBF-PS method is applied for the numerical solution of the good Boussinesq equation. The numerical method is based on scattered data interpolation using radial basis functions. The scheme is tested for single soliton, collision of breathers and soliton doublets. The results obtained from the method are compared with the exact solutions and the earlier work.

Mathematics Subject Classification: 65M20

Keywords: RBF, sine-Gordon equation, RBF interpolation method

1 Introduction

The radial basis function scheme was introduced by Kansa [7] to solve partial differential equations. It was shown by Micchelli in 1986 that for distinct interpolation points the interpolation system obtained in the multiquadric (MQ) method is always uniquely solvable. In the past two decade RBF interpolation methods have received increased attention for numerically solving partial differential equations (PDEs) on irregular domains by a global collocation approach (see, e.g., Kansa [8], Hon and Mao [6], Fasshauer [4], Larsson and Fornberg [9]). When proper attention is paid to boundaries, these methods

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can be spectrally accurate, and they generally result in having to solve a large, often ill-conditioned, dense linear system. The RBFs scheme is truly a mesh-free method which does not require the generation of a mesh, and since the MQ is infinitely differentiable, we can approximate the higher-order spatial derivatives that appear in the PDE directly by computing the corresponding derivatives of the basis functions.

In this work, we use an RBF-PS method [5] for the numerical solution of the good Boussinesq equation. The good Boussinesq equation, a nonlinear equation which describes shallow water waves propagating in both directions, is given by

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + q\frac{\partial^4 u}{\partial x^4}(x,t) + \frac{\partial^2 (u^2)}{\partial x^2}(x,t), \quad (x,t) \in [a,b] \times [0,T], \tag{1}$$

with the initial conditions and boundary conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \tag{2a}$$

$$u(a,t) = g_1(t),\ u(b,t) = g_2(t),\ u_x(a,t) = g_3(t),\ u_x(b,t) = g_4(t). \tag{2b}$$

Here $q = \pm 1$ is a real parameter. The value $q = -1$ leads to the good Boussinesq or well-posed equation (see [3] and the references therein), whereas for $q = 1$ one gets the bad Boussinesq or ill-posed equation.

1.1 RBF-PS method for good Boussinesq equation

We transform the good Boussinesq equation into a system of coupled first-order (in time) equations given by

$$\frac{\partial u}{\partial t}(x,t) = v(x,t), \tag{3a}$$

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + q\frac{\partial^4 u}{\partial x^4}(x,t) + \frac{\partial^2 (u^2)}{\partial x^2}(x,t), \qquad (x,t) \in [a,b] \times [0,T], \tag{3b}$$

with the boundary conditions and initial conditions

$$u(a,t) = g_1(t), u(b,t) = g_2(t), v(a,t) = f_1(t), v(b,t) = f_2(t), t \geq 0, \tag{4a}$$

$$u(x,0) = u_0(x), v(x,0) = u_1(x), x \in [a,b]. \tag{4b}$$

For simplicity we use a first-order Euler time integration scheme to deal with the temporal derivatives, so from the system (3) we can write

$$\left[\frac{U^{n+1}(x)-U^n(x)}{\Delta t}\right] = V^n(x), \tag{5a}$$

$$\left[\frac{V^{n+1}(x)-V^n(x)}{\Delta t}\right] = U^n_{xx}(x) + qU^n_{xxxx}(x) + 2U^n(x)U^n_{xx}(x) + 2(U^n_x(x))^2, \tag{5b}$$

Here Δt is the time step size and u^n (n being a non-negative integer) is the approximate solution at time $t^n = n\Delta t$. A higher-order time integrator could also be used. Re-arranging the equations in (5) we get

$$U^{n+1}(x) = U^n(x) + \Delta t V^n(x), \tag{6a}$$

$$V^{n+1}(x) = V^n(x) + \Delta t \left(U^n_{xx}(x) + qU^n_{xxxx}(x) + 2U^n(x)U^n_{xx}(x) + 2(U^n_x)^2(x)\right). \tag{6b}$$

The RBF approximations at time $t^n = n\Delta t$ of the solutions u and v of the two equations in (3) are given by

$$U^n(x) = \sum_{j=1}^{N} \lambda^n_{1j}\phi(\|x-x_j\|), \quad V^n(x) = \sum_{j=1}^{N} \lambda^n_{2j}\phi(\|x-x_j\|). \tag{7}$$

By enforcing the equations in (6) along with the boundary conditions given in equation (4) at the collocation points $\{x_i\}_{i=1}^N$, we obtain a system of time-stepping equations that can be written in vector form as

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \mathbf{V}^n, \tag{8a}$$

$$\mathbf{V}^{n+1} = \mathbf{V}^n + \Delta t \left(\mathbf{U}^n_{xx} + q\mathbf{U}^n_{xxxx} + 2\mathbf{U}^n .* \mathbf{U}^n_{xx} + 2\mathbf{U}^n_x .* \mathbf{U}^n_x\right), \tag{8b}$$

where $\mathbf{U}^n = [U^n(x_i)]_{i=1}^N$ and $.*$ denotes element-wise product of vectors. The update of the derivative vectors is performed via differentiation matrices, e.g.,

$$\mathbf{U}^n_x = \mathbf{D}_x \mathbf{U}^n,$$

where $\mathbf{D}_x$ is obtained by first forming $\mathbf{D}_x = \mathbf{A}_x\mathbf{A}^{-1}$ and then adding boundary conditions by appropriately replacing rows 1 and N of $\mathbf{D}_x$. The matrix $\mathbf{A}$ has entries $\phi(r_{ij})$ and the matrix $\mathbf{A}_x$ has entries $\frac{d}{dx}\phi(\|x_i - x\|)|_{x-x_j}$, $i, j = 1, \ldots, N$. Note that the system (8) requires no linear solves. The differentiation matrices $\mathbf{D}_x, \mathbf{D}_{xx}, \mathbf{D}_{xxxx}$ can be computed outside the time-stepping loop and this will be much faster.

2 Numerical examples

In this section, we apply the proposed method for the numerical solution of the good Boussinesq (GB) equation. The accuracy of the meshfree method is tested in terms of L_2, L_∞ error norms and the conservation of energy $M(t) = \int_a^b u(x,t)\, dx$, [3] of the GB equation.

2.1 Problem 1. Single soliton:

We consider the GB equation (1) as a system of two equations given in (3). The exact solution [3] of the equations in (3) are given as

$$u(x,t) = -\alpha\ \mathrm{sech}^2\left(\sqrt{\frac{\alpha}{6}}(x - x_0 - Ct)\right) - \left(\beta + \frac{1}{2}\right), \tag{9}$$

$$v(x,t) = -2\alpha\ C\ \sqrt{\frac{\alpha}{6}}\ \mathrm{sech}^2\left(\sqrt{\frac{\alpha}{6}}(x - x_0 - Ct)\right) \tanh\left(\sqrt{\frac{\alpha}{6}}(x - x_0 - Ct)\right),$$

$$C = \pm[-2(\beta + \alpha/3)]^{1/2}.$$

We solved the problem over the spatial domain $-40 \le x \le 80$ when $\Delta t = 0.0002$, $N = 241$. In our computations we used three types of radial basis functions, the multiquadric (MQ) $\phi(r) = \sqrt{c^2 + r^2}$ with shape parameter c, the Gaussian (GA) $\phi(r) = \exp(-cr^2)$ with shape parameter c and the quintic spline basis $\phi(r) = r^5$. The results are given in Tables 1-4 and Figure 1. In comparison the present method performs better than the methods given in references [1, 2, 3].

2.2 Problem 2. Two soliton interaction:

We consider the following initial conditions

$$u(x,0) = -\sum_{k=1}^{2}\left[\alpha_k\ \mathrm{sech}^2\left(\sqrt{\frac{\alpha_k}{6}}(x - \xi_k)\right) - \left(\beta_k + \frac{1}{2}\right)\right], \tag{10}$$

$$v(x,0) = -2\sum_{k=1}^{2}\alpha_k\ C_k\ \sqrt{\frac{\alpha}{6}}\ \mathrm{sech}^2\left(\sqrt{\frac{\alpha_k}{6}}(x - \xi_k)\right) \tanh\left(\sqrt{\frac{\alpha_k}{6}}(x - \xi_k)\right),$$

$$C_k = \pm[-2(\beta_k + \alpha_k/3)]^{1/2}, k = 1, 2.$$

Table 1: Error norms and energy constant for single soliton, when $C = 0.868332$, $\alpha = 0.369$ $\beta = -0.5$ in $[-40, 80]$, corresponding to Problem 1.

	t	1.2	3.6	9	36	72
MQ($c = 2$)	L_∞	3.654e-006	5.775e-006	3.783e-006	4.340e-006	2.205e-005
	L_2	9.583e-006	1.908e-005	1.844e-005	2.109e-005	7.975e-005
	M(t)	2.96356	2.96356	2.96357	2.96362	2.96340
	Amp.	0.36896	0.36864	0.36822	0.36769	0.36900
GA($c = 1$)	L_∞	3.658e-006	5.783e-006	3.823e-006	4.229e-006	4.574e-005
	L_2	9.515e-006	1.885e-005	1.780e-005	1.947e-005	1.107e-004
	M(t)	2.96356	2.96356	2.96356	2.96358	2.96330
	Amp.	0.36896	0.36864	0.36822	0.36769	0.36899
Ref.[1]	L_∞		0.920e-001	0.943e-001		
Ref.[2]	L_∞	0.269e-002			0.141e+000	0.130e+000
	L_2	0.370e-002			0.251e+000	0.323e+000
Ref.[3]	L_∞				0.103e-003	0.146e-003

Table 2: Error norms for single soliton for different values of α and C, when MQ($c = 2$), GA($c = 1$), $\beta = -0.5$ in $[-40, 80]$ corresponding to Problem 1.

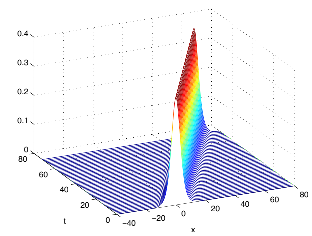
α	C	t	L_∞(MQ)	L_∞(GA)	L_∞([3])
1.2	0.44721	67.7	1.012e-002	9.674e-003	blow-up
1.5	0	20.7	1.576e-003	5.630e-006	blow-up

Table 3: Error norms for single soliton for different values of α and C, when MQ($c = 2$), GA($c = 1$), $\beta = -0.5$ at time $t = 1$, corresponding to Problem 1.

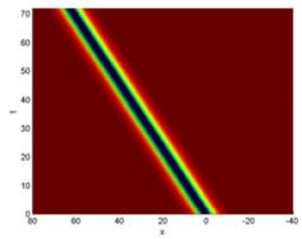
α	C	L_∞(MQ)	L_∞(GA)	$L_\infty(r^5)$
0.15	0.94868	6.292e-007	6.294e-007	3.290e-004
0.5	0.81650	5.169e-006	5.176e-006	1.149e-002
1.2	0.44721	7.170e-006	7.295e-006	1.375e-001
1.5	0	1.134e-006	1.382e-008	2.419e-001

Table 4: Error norms and energy constant versus time step size Δt for single soliton, when MQ($c = 2$), GA($c = 1$), $\beta = -0.5$, $\alpha = 0.369$, $C = 0.868332$ at time $t = 1$,corresponding to Problem 1.

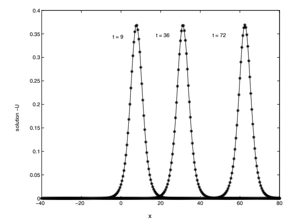
Δt	L_∞(MQ)	L_∞(GA)	$\lvert M(t) - M(0)\rvert$
0.01	1.571e-004	1.571e-004	0.00004
0.001	1.572e-005	1.572e-005	0.00004
0.0001	1.578e-006	1.581e-006	0.00004
0.00001	5.746e-007	1.669e-007	0.00004



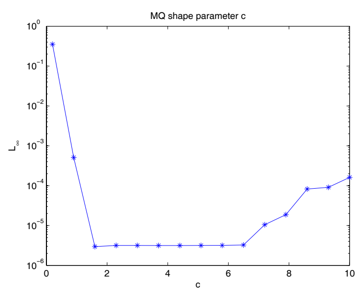
(a) Single soliton: approximate solution u



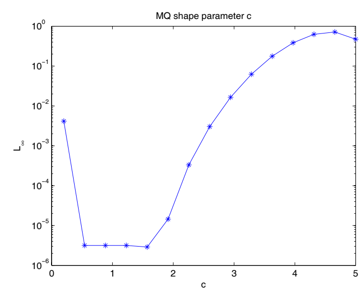
(b) Single soliton: approximate solution u



(c) Exact u (full curve), approximate u (dotted line)



(d) L_∞ error norm against MQ(c)



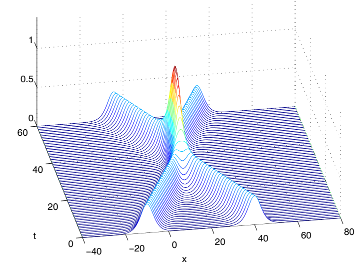
(e) L_∞ error norm against GA(c)

Figure 1: Single soliton: when $\Delta t = 0.0002$, $N = 241$, $c = 2$, $C = 0.868332$, $\alpha = 0.369$ $\beta = -0.5$ in $[-40, 80]$, corresponding to problem 1.

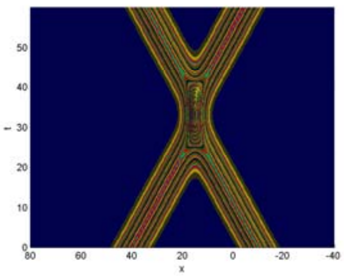
The above initial conditions are the sum of two solitary waves initially centered at $\xi_1 = -10$ and $\xi_2 = 40$ with the amplitudes α_1 and α_2. The two waves move toward each other with the speeds C_1 and C_2, respectively. In Figs. 2, the interaction of two waves with equal and unequal amplitudes are shown. The interaction of the two waves is elastic, and after interaction the waves retain their shape and amplitudes as shown in Fig. 2.

3 Conclusions

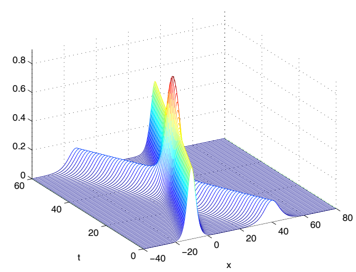
In this paper, an RBF approximation method is applied for the numerical solution of the good Boussinesq equation. We split the second-order in time problem into a system of two first-order equations. The technique used in this paper provides an efficient alternative for the solution of higher-order PDEs in time as well as in space. From an application viewpoint the implementation of this method is very simple and straightforward.



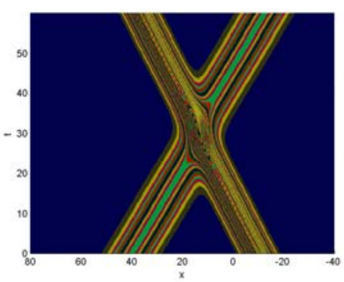
(a) Interaction of two waves with $\alpha_1 = \alpha_2 = 0.369$



(b) Interaction of two waves with $\alpha_1 = \alpha_2 = 0.369$



(c) Interaction of two waves with $\alpha_1 = 0.5, \alpha_2 = 0.15$



(d) Interaction of two waves with $\alpha_1 = 0.5, \alpha_2 = 0.15$

Figure 2: When $\Delta t = 0.0004$, MQ($c = 2$), $C_1 = 0.868332$, $C_2 = -0.868332$, $\beta_1 = -0.5$, $\beta_2 = -0.5$, corresponding to Problem 2.

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Table 5: Energy constant for two soliton interaction, when $\Delta t = 0.0004$, $N = 241$, MQ($c = 2$), GA($c = 1$), $C_1 = 0.868332$, $C_2 = -0.868332$, $\alpha_1 = 0.369$, $\alpha_2 = 0.369$ $\beta_1 = -0.5$, $\beta_2 = -0.5$, corresponding to Problem 2.

	MQ	GA
t	$\|M(t)\|$	$\|M(t)\|$
0	5.9518	5.9518
35	5.9521	5.9520
50	5.9523	5.9544

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