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Using a Super Efficiency Model for Ranking Units in DEA

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Abstract

This paper discusses about ranking decision making units (DMUs) in data envelopment analysis (DEA). We provide a input-output orientation super efficiency model, where our proposed model can be used in ranking the performance of DMUs.

Keywords: Data envelopment analysis, Linear programming, Decision making units, Super efficiency, Ranking.

1 Introduction

Data envelopment analysis (DEA) is a mathematical method for determining the relative efficiency of decision making units (DMUs). The first DEA model introduced by Charnes et al [3]; (called CCR model). DEA identifies an efficient frontier where all DMUs have a unity score. We know that usually there are DMUs which have this efficient frontier. Hence, ranking frontier DMUs is very important in the DEA.

Super efficiency DEA models can be used in ranking the performance of efficient DMUs. In fact, when a DMU under evaluation is not included in the reference set of the original DEA models, the resulting DEA models are called super efficiency DEA models. Then, the super efficiency DEA model obtained either situations constant return to scale (CRS) or value return to scale (VRS).

The CCR super efficiency DEA model is developed under CRS by Andersen and Petersen et al [1]; (called AP model). Thrall pointed out that the AP model may result in infeasibility and instability when some inputs are close to zero [5].

Similarly, Zhu showed that super efficiency DEA models with (CRS) could occur infeasibility if and only if a zero in data [7].
When consider super efficiency DEA model based upon the BCC model (VRS super efficiency model), then infeasibility of the related linear program is very likely to occur (see [2,4]). Seiford and Zhu show the necessary and sufficient conditions of infeasibility in VRS super efficiency model. Yao argues that super efficiency can be interpreted as input saving and output surplus achieved by an efficient DMU [6]. By utilizing this interpreted, we proposed a VRS super efficiency model with input [input saving] output [output surplus] orientation, which shows this model can be feasible always.
The paper is organized as follows. Section 2 discusses issues related to the VRS super efficiency DEA model. The input-output orientation super efficiency model is introduced in Section 3 and Section 4 concludes the paper.

2 VRS super efficiency DEA model

Suppose, we have n DMUs where each DMUj , j = 1, ..., n, produces s output, y_{rj} (r = 1, ..., s) using m input, x_{ij} (i = 1, ...,m). The efficiency of a specific DMU$_o$, $o \in \{1,2,...n\}$ can be evaluated by the BCC model. This model is as follows:

$$
\begin{aligned}
&Min \quad \theta_o \\
&s.t \quad \sum_{j=1}^{n} \lambda_j x_{ij} - \theta_o x_{io} \le 0 \qquad i = 1,2,...,m \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} \qquad r = 1,2,...,s \qquad (1) \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1 \\
&\qquad \lambda_j \ge 0 \qquad j = 1,2,...,n
\end{aligned}
$$

Where x_{io} and y_{ro} are respectively the i^{ih} input and r^{ih} output for a DMU $_o$ under evaluation. We suppose that θ^* shows optimal amounts. θ_o^* represents efficiency score and all the frontier DMUs have θ_o^*=1. In order to determining the performance of frontier DMUs, we use the super efficiency DEA model. The VRS super efficiency DEA model related to BCC model can be expressed as (Seiford and Zhu, [4]):

$$
\begin{aligned}
&\text{Min} \quad \theta_o \\
&\text{s.t} \quad \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} - \theta_o x_{io} \leq 0 \qquad i = 1,2,\ldots,m \\
&\qquad \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1,2,\ldots,s \qquad (2) \\
&\qquad \sum_{j=1, j\neq o}^{n} \lambda_j = 1 \\
&\qquad \lambda_j \geq 0 \qquad j = 1,2,\ldots,n,\ j \neq o
\end{aligned}
$$

Suppose that all data are positive, Seiford and Zhu show that model (2) is infeasible if and only if $g^* < 1$, where g^* is the optimal solution to the following model:

$$
\begin{aligned}
&Min \quad g \\
&s.t \quad \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \geq g y_{ro} \qquad r = 1,2,\ldots,s \qquad (3) \\
&\qquad \sum_{j=1, j\neq o}^{n} \lambda_j = 1 \\
&\qquad \lambda_j \geq 0 \qquad j = 1,2,\ldots,n,\ j \neq o
\end{aligned}
$$

Thus, if DMU_o has the largest output values, model (2) becomes infeasible and in order to obtain a complete ranking, we have to solve the infeasibility problem associated with model (2).

3 The super efficiency model with input-output orientation

In this section, we introduce a super efficiency model in the input-output oriented.

Suppose that $\text{DMU}_o = (x_o, y_o)$ is unit under evaluation. The model is as:

$$
\begin{aligned}
&Min \quad 1+\beta_o \\
&s.t \quad \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} - (1+\beta_o) x_{io} \leq 0 \qquad i = 1,2,...,m \\
&\qquad \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} - (1-\beta_o) y_{ro} \geq 0 \qquad r = 1,2,...,s \qquad (4) \\
&\qquad \sum_{j=1, j\neq o}^{n} \lambda_j = 1 \\
&\qquad \lambda_j \geq 0 \qquad j = 1,2,...,n,\ j \neq o
\end{aligned}
$$

We have the following two propositions:

Proposition 1: Let (x_k, y_k) and (x_j, y_j) be the two DMUs in the proposed model with $x_k \leq x_j$ and $y_k \geq y_j$. The objective values of the proposed model corresponding to (x_k, y_k) will be no less then that corresponding to (x_j, y_j).

Proof: It is obvious.

Proposition 2: Suppose that all data are positive then proposed model is always feasible.

Proof: Suppose that propose model is infeasible (Contrary suppose). We consider dual it which follows:

$$
\begin{aligned}
&Max \quad 1 + Uy_o - Vx_o + u_o \\
&s.t \quad Uy_j - Vx_j + u_o \leq 0 \qquad j = 1,2,...,n,\ j \neq o \\
&\qquad Uy_o + Vx_o = 1 \qquad (5) \\
&\qquad U, V \geq 0
\end{aligned}
$$

The model above is feasible. Because

$$U_r = 0(\forall r), V_k = 1/x_{ko} \ (if \ x_{ko} \neq 0), V_i = 0(\forall i \neq r)$$

is a feasible solution. So propose model can not be unbounded. If the model (4) is unbounded then we must have:

$$\exists d = (d_u, d_v, d_{uo}) \neq 0$$

$$d_u y_o - d_v x_o + d_{uo} > 0 \qquad (I)$$

$$d_u y_j - d_v x_j + d_{uo} \leq 0 \qquad (II)$$

$$d_u y_o + d_v x_o \geq 0 \qquad (III)$$

$$d_u, d_v \geq 0, d_{uo} \ \ is\ free \ \ (IV)$$

Using (III) and (IV) we can get $d_u, d_v = 0$. Then

$$(I) \Rightarrow d_{uo} > 0$$

$$(II) \Rightarrow d_{uo} \leq 0$$

Therefore, the contrary suppose is false, and the proof is complete.

4 Conclusions

In this paper, a novel ranking method for decision making units (DMUs) is discussed in data envelopment analysis (DEA). An input-output orientation super

efficiency model is provided and the proposed method is used to ranking the performance of DMUs.

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