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Solution and Stability Analysis of a Mathematical Model of Four Species Syn - Eco Symbiosis

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Abstract

The mathematical model equations characterizing the syn eco system constitute a set of four first order non linear coupled differential equations. In this paper, series solutions are obtained for the four species syn eco-symbiosis model which is governed by a system of non linear equations. For this purpose, the Adomian decomposition method (ADM) and Homotopy perturbation method (HPM) are employed. Also, it is shown that the above system of equations is globally asymptotically stable under certain conditions. Another important feature of the paper is that Adomian polynomials are incorporated in to the HPM to evaluate solutions making the method much easier.

Mathematics Subject Classifications: 34A34, 34B15, 34D10, 37M99

Keywords: syn eco-symbiosis, Adomian decomposition, Homotopy perturbation, stability

1. Introduction

Nonlinear differential equations play a crucial role in Applied mathematics and Engineering. Over the last ten years, many mathematical methods aimed at obtaining analytical solutions of nonlinear differential equations arising in various fields of science and engineering have appeared in the research literature. In this paper, Adomian decomposition method and Homotopy perturbation methods are employed to obtain series solutions to four species syn-eco symbiosis model which is governed by a system of non linear differential equations. We show that the solutions obtained by both ADM and HPM are numerically same by giving an example. We also show that the positive equilibrium of the above model is globally asymptotically stable. The main feature of the methods is that Adomian polynomials are incorporated into Homotopy perturbation method to evaluate the series solutions.

2. Analysis of the model by Adomian Decomposition method

The four species syn eco symbiosis model is governed by a system of non linear equations. The equations are

$x_1' = x_1(a_1 - b_1x_1 - c_1x_2 + d_1x_3)$
$x_2' = x_1(a_2 + b_2x_1 - c_2x_2)$
$x_3' = x_3(a_3 - c_3x_3 + d_3x_4)$
$x_4' = x_4(a_4 + c_4x_3 - d_4x_4)$ (2.1)

where, $a_i\ (i = 1,2,3,4)$, $b_i\ (i = 1,2)$, $c_i\ (i = 1,2,3,4)$, $d_i\ \ (i = 1,3,4)$ are constants. Here x_i refers to the population of prey - predator species and a_i are natural growth rates of x_i. $b_1, c_2, c_3\ and\ d_4$ are self inhibition coefficients of $x_{i\,.}$ c_1, b_2 are interaction coefficients of $x_1\ due\ to\ x_2\ and\ x_2\ due\ to\ x_1$. d_1 is the coefficient for commensal for $x_1\ due\ to\ x_3$. $d_3,\ c_4$ are mutual interaction coefficients between $x_3 and\ x_4$. $\frac{a_1}{b_1}, \frac{a_2}{c_2}, \frac{a_{3,}}{c_3}, \frac{a_4}{c_4}$ are carrying capacities of $x_i\ (i = 1,2,3,4)$ respectively and all $x_i \geq 0$ and all $a_i,\ b_i, c_i, d_i \geq 0$. Though the system has sixteen equilibrium states ,we assume only the positive equilibrium state obtained by making $\frac{dx_i}{dt} = 0\ , i = 1,2,3,4$.

$\alpha_1 = \frac{p+c_2d_1q}{r}$, $\alpha_2 = \frac{s+c_2b_2q}{r}$, $\alpha_3 = \frac{a_4d_3+a_3d_{4,}}{c_3d_4-c_4d_{3,}}$, $\alpha_4 = \frac{a_4c_3+a_3c_4}{c_3d_4-c_4d_3}$ where

$p = (a_1c_2 + a_2c_1)(c_3d_4 - c_4d_3)$
$q = (a_4d_3 + a_3d_4)$
$r = (b_1c_2 + b_2c_1)(c_3d_4 - c_4d_3)$
$s = (a_1b_2 - a_2b_1)\ (c_3d_4 - c_4d_3)$ (2.2)

For the existence of positive equilibrium, we assume $\frac{c_3}{c_4} > \frac{d_3}{d_4}\ \ and\ \ \frac{a_1}{a_2} > \frac{b_1}{b_2}$.

Defining $u(\tau) = \frac{x_1}{\alpha_1}, v(\tau) = \frac{x_2}{\alpha_2}\ , w(\tau) = \frac{x_3}{\alpha_3}\ , z(\tau) = \frac{x_4}{\alpha_4}$ at $\tau = a_1t,$ (2.3)

the system of equations (1.1) are transformed to

$\frac{du(\tau)}{d\tau} = u(\tau) - \ k_0u^2(\tau) - k_1u(\tau)v(\tau) + k_2u(\tau)w(\tau)$
$\frac{dv(\tau)}{d\tau} = k_3v(\tau) + k_4u(\tau)v(\tau) - k_5v^2(\tau)$
$\frac{dw(\tau)}{d\tau} = k_6w(\tau) - k_7w^2(\tau) + k_8z(\tau)w(\tau)$

$\frac{dz(\tau)}{d\tau} = k_9 z(\tau) + k_{10} w(\tau) z(\tau) - k_{11} z^2(\tau)$ (2.4)

where,

$k_0 = \frac{b_1}{a_1}\alpha_1,\ k_1 = \frac{c_1}{a_1}\alpha_2,\ k_2 = \frac{d_1}{a_1}\alpha_3,\ k_3 = \frac{a_2}{a_1},\ k_4 = \frac{b_2}{a_1}\alpha_1,\ k_5 = \frac{c_2}{a_1},$

$k_6 = \frac{a_3}{a_1},\ k_7 = \frac{c_3}{a_1}\alpha_3,\ k_8 = \frac{d_3}{a_1}\alpha_4,\ k_9 = \frac{a_4}{a_1},\ k_{10}\frac{c_4}{a_1}\alpha_3,\ k_{11} = \frac{d_4}{a_1}\alpha_4$

Introducing the differential operator $L = \frac{d}{dt}$, the system (2.4) transforms to

$Lu(\tau) = u(\tau) - k_0 f(u) - k_1 \Phi_1\big(u(\tau)v(\tau)\big) + k_2 \Phi_2(u(\tau)w(\tau))$

$Lv(\tau) = k_3 v(\tau) + k_4 \Phi_1\big(u(\tau)v(\tau)\big) - k_5 g(v(\tau))$

$Lw(\tau) = k_6 w(\tau) - k_7 h\big(w(\tau)\big) + k_8 \Phi_3(z(\tau), w(\tau))$

$Lz(\tau) = k_9 z(\tau) + k_{10}\Phi_3\big(z(\tau), w(\tau)\big) - k_{11} i(z(\tau))$ (2.5)

with initial conditions $u(0) = l_1,\ v(0) = l_2,\ w(0) = l_3,\ z(0) = l_4.$ (2.6)

Here $f, g, h, i, \Phi_1, \Phi_2, \Phi_3$ are nonlinear functions of $u, v, w, z, (u,v), (u,w), (z,w)$ respectively. The decomposition method consists of approximating the solutions of (2.5) as an infinite series.

$u = \sum_{n=0}^{\infty} u_n$, $f(u) = u^2 = \sum_{n=0}^{\infty} A_n$ where $A_n = \sum_{k=0}^{n} u_k u_{n-k}$

$v = \sum_{n=0}^{\infty} v_n$ $g(v) = v^2 = \sum_{n=0}^{\infty} B_n$ where $B_n = \sum_{k=0}^{n} v_k v_{n-k}$

$w = \sum_{n=0}^{\infty} w_n$ $h(w) = w^2 = \sum_{n=0}^{\infty} C_n$ where $C_n = \sum_{k=0}^{n} c_k c_{n-k}$

$z = \sum_{n=0}^{\infty} z_n$, $i(z) = z^2 = \sum_{n=0}^{\infty} H_n$ where $H_n = \sum_{k=0}^{n} z_k z_{n-k}$ (2.7)

$\Phi_1(u,v) = uv = \sum_{n=0}^{\infty} D_n = (\sum_{n=0}^{\infty} u_n \sum_{n=0}^{\infty} v_n) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} u_k v_{n-k})$

or $D_n = u_k v_{n-k}$,n=0,1,2...

$\Phi_2(u,w) = uw = \sum_{n=0}^{\infty} E_n = (\sum_{n=0}^{\infty} u_n \sum_{n=0}^{\infty} w_n) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} u_k w_{n-k})$

$E_n = (u_k' w_{n-k})$ n=0, 1, 2...

$\Phi_3(z,w) = zw = \sum_{n=0}^{\infty} F_n = (\sum_{n=0}^{\infty} z_n \sum_{n=0}^{\infty} w_n) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} z_k w_{n-k})$

$F_n = (z_k' w_{n-k})$ n=0, 1, 2... (2.8)

Assume that inverse integration operator L^{-1} exists and is defined as $L^{-1} = \int_0^{\tau}(.)d\tau$. Applying inverse operator to (2.5) and introducing initial conditions u_0, v_0, w_0, z_0 and from (2.7), (2.8), we obtain

$u(\tau) = u_0 + L^{-1}\sum_{n=0}^{\infty} u_n - k_0 L^{-1}\sum_{n=0}^{\infty} A_n - k_1 L^{-1}\sum_{n=0}^{\infty} D_n + k_2 L^{-1}\sum_{n=0}^{\infty} E_n$

$v(\tau) = v_0 + k_3 L^{-1}\sum_{n=0}^{\infty} v_n + k_4 L^{-1}\sum_{n=0}^{\infty} D_n - k_5 L^{-1}\sum_{n=0}^{\infty} B_n$

$w(\tau) = w_0 + k_6 L^{-1}\sum_{n=0}^{\infty} w_n + k_7 L^{-1}\sum_{n=0}^{\infty} C_n - k_8 L^{-1}\sum_{n=0}^{\infty} F_n$

$z(\tau) = z_0 + k_9 L^{-1}\sum_{n=0}^{\infty} z_n + k_{10} L^{-1}\sum_{n=0}^{\infty} F_n - k_{11} L^{-1}\sum_{n=0}^{\infty} H_n$ (2.9)

Now determine the iterates using recurrence relation as

$u_0 = l_1,$

$u_{n+1} = L^{-1}u_n - k_0 L^{-1}A_n - k_1 L^{-1}D_n + k_2 L^{-1}E_n$, n=0, 1, 2…,

$v_0 = l_2,$

$v_{n+1} = k_3 L^{-1}v_n + k_4 L^{-1}D_n - k_5 L^{-1}B_n$, n=0, 1,2,…

$w_0 = l_3,$

$w_{n+1} = k_6 L^{-1}w_n - k_7 L^{-1}C_n + k_8 L^{-1}F_n$, n=0, 1,2,…

$z_0 = l_4,$
$z_{n+1} = k_9 L^{-1} z_n + k_{10} L^{-1} F_n - k_{11} L^{-1} H_n$, n=0, 1,2,... (2.10)
We write the solution of the initial value problem (2.5) as

$$(u, v, w, z) = \left\lfloor \lim_{n\to\infty} \sum_{k=0}^{n} u_k, \lim_{n\to\infty} \sum_{k=0}^{n} v_k, \lim_{n\to\infty} \sum_{k=0}^{n} w_k, \lim_{n\to\infty} \sum_{k=0}^{n} z_k \right\rfloor$$

Application of Homotopy Perturbation method

We use Homotopy perturbation method to model equations (2.4) as

$\frac{du(\tau)}{d\tau} = p(u(\tau) - k_0 u^2(\tau) - k_1 u(\tau)v(\tau) + k_2 u(\tau)w(\tau))$
$\frac{dv(\tau)}{d\tau} = p(k_3 v(\tau) + k_4 u(\tau)v(\tau) - k_5 v^2(\tau))$
$\frac{dw(\tau)}{d\tau} = p(k_6 w(\tau) - k_7 w^2(\tau) + k_8 z(\tau)w(\tau))$
$\frac{dz(\tau)}{d\tau} = p(k_9 z(\tau) + k_{10} w(\tau)z(\tau) - k_{11} z^2(\tau))$ (3.1)

where $p \in [0,1]$ is an embedding parameter. As in He's Homotopy perturbation method, when $p = 0$, it is a linear equation. When $p = 1$, it becomes the original nonlinear equation. We consider the embedding parameter as a small parameter. We assume the solutions of (3.1) as a power series by letting

$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots$
$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots$
$w = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots$
$z = z_0 + pz_1 + p^2 z_2 + p^3 z_3 + \cdots$ (3.2)

Substituting (3.2) in (3.1) and equating the coefficients of like powers of p, we obtain the following differential equations;

p^0: $u_0' = 0$
$v_0' = 0$
$w_0' = 0$
$z_0' = 0$ Where $u_0 = l_1,\ v_0 = l_2,\ w_0 = l_3,\ z_0 = l_4.$

p^1: $u_1' = u_0 - k_0 A_0 - k_1 D_0 + k_2 E_0$
$v_1' = k_3 v_0 + k_4 D_0 - k_5 B_0$
$w_1' = k_6 w_0 - k_7 C_0 + k_8 F_0$
$z_1' = k_9 z_0 + k_{10} F_0 - k_{11} H_0$

p^2: $u_2' = u_1 - k_0 A_1 - k_1 D_1 + k_2 E_1$
$v_2' = k_3 v_1 + k_4 D_1 - k_5 B_1$
$w_2' = k_6 w_1 - k_7 C_1 + k_8 F_1$

$$z_2' = k_9 z_1 + k_{10} F_1 - k_{11} H_1$$
$$p^3: \quad u_3' = u_2 - k_0 A_2 - k_1 D_2 + k_2 E_2$$
$$v_3' = k_3 v_2 + k_4 D_2 - k_5 B_2$$
$$w_3' = k_6 w_2 - k_7 C_2 + k_8 F_2$$
$$z_3' = k_9 z_2 + k_{10} F_2 - k_{11} H_2$$
$$p^4: \quad u_4' = u_3 - k_0 A_3 - k_1 D_3 + k_2 E_3$$
$$v_4' = k_3 v_3 + k_4 D_3 - k_5 B_3$$
$$w_4' = k_6 w_3 - k_7 C_3 + k_8 F_3$$
$$z_4' = k_9 z_3 + k_{10} F_3 - k_{11} H_3$$
$$p^5: \quad u_5' = u_4 - k_0 A_4 - k_1 D_4 + k_2 E_4$$
$$v_5' = k_3 v_4 + k_4 D_4 - k_5 B_4$$
$$w_5' = k_6 w_4 - k_7 C_4 + k_8 F_4$$
$$z_5' = k_9 z_4 + k_{10} F_4 - k_{11} H_4 \quad (3.3)$$

where $A_n, B_n, C_n, D_n, E_n, F_n, H_n, n = 0,1,2,3,4$ are the Adomian polynomials (2.7), (2.8) respectively. These polynomials are incorporated into the Homotopy perturbation method to evaluate solution of the method in a much easier way.

3. Stability analysis of four species syn - ecosymbiosis

Theorem: Suppose the hypothesis $b_1 + c_3 > c_2 + d_1 + d_4$ holds true. Then, the positive equilibrium α_i $(i = 1,2,3,4)$ of the system (1.1) is globally asymptotically stable.

Proof: Consider the transformation

$y_1 = x_1 - \alpha_1, y_2 = x_2 - \alpha_2, \; y_3 = x_3 - \alpha_3, \; y_4 = x_4 - \alpha_4.$

Then equations (1.1) lead to

$$y_1' = -(y_1 + \alpha_1)(b_1 y_1 + c_1 y_2 - d_1 y_3)$$
$$y_2' = (y_2 + \alpha_2)(b_2 y_1 - c_2 y_2)$$
$$y_3' = -(y_3 + \alpha_3)(c_3 y_3 - d_3 y_4)$$
$$y_4' = (y_4 + \alpha_4)(c_4 y_4 - d_4 y_4) \quad (4.1)$$

Select a scalar function of the type

$$V(y_1, y_2, y_3, y_4) = V(y_i) = \sum_{i=1}^{4} \int_0^{y_i} \frac{s ds}{s + \alpha_i} \; (i = 1,2,3,4) \quad (4.2)$$

It is clear that $V(y_i) > 0$ if $y_i(0) + \alpha_i > 0$ for (i = 1,2,3,4) and $V(0) = 0$. Hence, it is positive definite on

$\Omega = \{y_i \,; \; y_i + \alpha_i > 0, i = 1,2,3,4\}$

Then the time derivative V along with the solutions of (4.1) is given by

$$V^*(y_i) = \sum_{i=1}^{4} \frac{y_i y_i'}{y_i + \alpha_i} = -b_1 y_1^2 - c_2 y_2^2 - c_3 y_3^2 - d_4 y_4^2 - (c_1 - b_2) y_1 y_2 + (d_3 + c_4) y_3 y_4 + d_1 y_1 y_3 \quad (4.3)$$

From the fact that
$y_1^2 + y_2^2 \geq 2y_1y_2$
$y_3^2 + y_4^2 \geq 2y_3y_4$
$y_1^2 + y_3^2 \geq 2y_1y_3$, we obtain
$V^*(y_i) \leq -b_1y_1^2 - c_2y_2^2 - c_3y_3^2 - d_4y_4^2 - (c_1 - b_2)(\frac{y_1^2+y_2^2}{2})$ $+(d_3 + c_4)(\frac{y_3^2+y_4^2}{2})$ $+d_1(\frac{y_1^2+y_3^2}{2})$
$=-\frac{y_1^2}{2}(2b_1 + c_1 - d_1 - b_2) - \frac{y_2^2}{2}(2c_2 - b_2 + c_1) =-\frac{y_3^2}{2}(2c_3 - d_3 - c_4 - d_1)$ -
$\frac{y_4^2}{2}(2d_4 - c_4 - d_3)$ (4.4)
⇒ Using the condition $b_1 + c_3 > c_2 + d_1 + d_4$, $V^*(y_i)$ is negative definite on Ω.
Apply the results (1) and (2) that follow.
(1) "Assume that there exists a scalar function $V(x)$ such that (1) V is positive definite on R^n and $V(x) \to \infty$ as $\|x\| \to \infty$ and
(2) $V^*(x)$ is negative definite at all points $x\epsilon R^n$, then the zero solution of $x' = f(x)$ is globally asymptotically stable",
We find that the equilibrium $\alpha_i(i = 1,2,3,4)$ is globally asymptotically stable.

5. An example for numerical comparison between Adomian decomposition method and Homotopy perturbation method

5.1 Adomian decomposition method

Letting $a_1 = a_2 = a_3 = a_4 = 1$, $b_1 = 1$, $b_2 = 2$, $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$, $c_3 = 2$, $c_4 = 1$, $d_1 = d_3 = d_4 = 1$ in (2.1),
and $u_0 = v_0 = w_0 = 1$, $z_0 = 2$ as initial approximations, the Adomian decomposition iterations are obtained as
$u_0 =1$, $v_0=1$, $w_0 = 1$, $z_0 = 2$
$u_1 = \frac{1}{2}\tau$, $v_1 = 4\tau$, $w_1 = 3\tau$, $z_1 = -6\tau$
$u_2 = 1.875\,\tau^2$, $v_2 = 7.875\tau^2$, $w_2 = -10.5\,\tau^2$, $z_2 = 33\,\tau^2$
$u_3 = -9.1093\,\tau^2$, $v_3 = 10.238\tau^2$, $w_3 = 6.5\,\tau^2$, $z_3 = -161\,\tau^2$
$u_4 = 6.058\,\tau^2$, $v_4 = 1.9368\tau^2$, $w_4 = 62.125\,\tau^2$, $z_4 = 746.75\,\tau^2$
The rest of the terms of the decomposition series have been calculated using Mathcad7.
Substituting these terms in (2.7), we get the following results
$u(\tau) = u_0(\tau) + u_1(\tau) + u_2(\tau) + u_3(\tau) + u_4(\tau) + \cdots$

$u(\tau) = 1 + 0.5\tau + 1.875\ \tau^2 - 9.1093\ \tau^2 + 6.058\ \tau^2 + ..$
$v(\tau) = v_0(\tau) + v_1(\tau) + v_2(\tau) + v_3(\tau) + v_4(\tau) + \cdots$
$v(\tau) = 1 + 4\tau + 7.875\tau^2 + 10.238\tau^2 + 1.9368\tau^2 + \ldots$
$w(\tau) = w_0(\tau) + w_1(\tau) + w(\tau) + w_3(\tau) + w_4(\tau) + \cdots$
$w(\tau) = 1 + 3\tau - 10.5\ \tau^2 + 6.5\ \tau^2 + 62.125\ \tau^2 + \ldots$
$z(\tau) = z_0(\tau) + z_1(\tau) + z_2(\tau) + z_3(\tau) + z_4(\tau) + \cdots$
$z(\tau) = 2 - 6\tau + 33\ \tau^2 - 161\ \tau^2 + 746.75\ \tau^2 - \ldots$ (5.1)

5.2 Homotopy perturbation method

Solving the system of equations (3.3) by taking the same values as in the above example, we obtain

$u_0 = 1,\ v_0 = 1,\ w_0 = 1,\ z_0 = 2$
$u_1 = \frac{1}{2}\ \tau,\ v_1 = 4\tau,\ w_1 = 3\tau,\ z_1 = -6\tau$
$u_2 = 1.875\ \tau^2,\ v_2 = 7.875\tau^2,\ w_2 = -10.5\ \tau^2,\ z_2 = 33\ \tau^2$
$u_3 = -9.1093\ \tau^2,\ v_3 = 10.238\tau^2,\ w_3 = 6.5\ \tau^2,\ z_3 = -161\ \tau^2$
$u_4 = 6.058\ \tau^2,\ v_4 = 1.9368\tau^2,\ w_4 = 62.125\ \tau^2,\ z_4 = 746.75\ \tau^2$

Substituting u_i, v_i, w_i, z_i(i=1, 2, 3, 4) in to(4.2),we have
$u(\tau) = 1 + 0.5p\tau + 1.875\ p^2\tau^2 - 9.1093\ p^3\tau^2 + 6.058p^4\ \tau^2 + \ldots$
$v(\tau) = 1 + 4p\tau + 7.875p^2\tau^2 + 10.238p^3\tau^2 + 1.9368p^4\tau^2 + \ldots$
$w(\tau) = 1 + 3p\tau - 10.5p^2\ \tau^2 + 6.5p^3\ \tau^2 + 62.125p^4\ \tau^2 + \ldots$
$z(\tau) = 2 - 6p\tau + 33p^2\ \tau^2 - 161\ p^3\tau^2 + 746.75\ p^4\tau^2 - \ldots$

Letting $p \to 1$,we obtain
$u(\tau) = 1 + 0.5\tau + 1.875\ \tau^2 - 9.1093\ \tau^2 + 6.058$
$v(\tau) = 1 + 4\tau + 7.875\tau^2 + 10.238\tau^2 + 1.9368\tau^2 + \ldots$
$w(\tau) = 1 + 3\tau - 10.5\ \tau^2 + 6.5\tau^2 + 62.125\ \tau^2 + \ldots$
$z(\tau) = 2 - 6\tau + 33\ \tau^2 - 161\ \tau^2 + 746.75\ \tau^2 - \ldots$ (5.2)

which are exactly the same solutions as those obtained in (5.1) using Adomian decomposition method.

6. Conclusions

In this paper, Adomian decomposition method and Homotopy perturbation method are employed to obtain series solutions to four species syn - eco symbiosis model which is governed by a system of non linear differential equations. We show that the solutions obtained by both the methods ADM and HPM are numerically same by giving an example. We also show that the positive equilibrium of the above model is globally asymptotically stable. The main feature of this paper is that Adomian polynomials are incorporated into Homotopy perturbation method to evaluate the series solutions.

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