Applied Mathematical Sciences, Vol. 6, 2012, no. 35, 1719 - 1725

Hyers-Ulam-Rassias Stability of a Cubic Functional Equation in RN-Spaces: A Direct Method

Afshin Erami

Department of Mathematics, Marvdasht Branch
Islamic Azad University, Marvdasht, Iran
eramiafshin@yahoo.com

Habib Hoseini

Department of Mathematics, Firoozabad Branch
Islamic Azad University, Firoozabad, Iran
mmmomid@gmail.com

Dong Yun Shin

Department of Mathematics, University of Seoul
Seoul 130-743, Republic of Korea
dyshin@uos.ac.kr

Hassan Azadi Kenary

Department of Mathematics, Yasouj University
Yasouj 75914-353, Iran
azadi@mail.yu.ac.ir

Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of the following cubic functional equation:

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y)$$

in random normed spaces: A direct method.

Mathematics Subject Classification: 39B82, 39B52

Keywords: Hyers-Ulam-Rassias stability, Random normed spaces

1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [11] in 1940. In the next year, Hyers [6] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [7] proved a generalization of Hyers's theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations.

2. Preliminaries

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [9].

Throughout this paper, let Γ^+ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Gamma^+ : l^- F(-\infty) = 1\}$, where $l^- f(x) = \lim_{t\to x^-} f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0 & \text{if} \quad t \leq a, \\ 1 & \text{if} \quad t > a. \end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(x,1) = x$ for all $x \in [0,1]$;
(d) $T(x,y) \le T(z,w)$ whenever $x \le z$ and $y \le w$ for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are as follows: $T(x,y) = xy$, $T(x,y) = \max\{a+b-1,0\}$, $T(x,y) = \min(a,b)$. Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^{n} x_i$ is defined recursively by $T_{i=1}^{1} x_1 = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \ge 2$. $T_{i=n}^{\infty} x_i$ is defined by $T_{i=1}^{\infty} x_{n+i}$.

Definition 2.2. A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:
(a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;
(b) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \ge 0$;
(c) $\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and $t,s \ge 0$.

Definition 2.3. Let (X,μ,T) be an RN-space.
(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.
(3) The RN-space (X,μ,T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 2.1. ([9]) *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.

Throughout this paper, using direct method we proved the generalized Hyers-Ulam stability of the following functional equation:

$$3f(x+3y) + f(3x-y) = 15f(x+y) + 15f(x-y) + 80f(y) \tag{2.1}$$

in random normed spaces.

3. **Random Stability of Functional Equation (2.1)**

In this section, we prove the generalized Hyers-Ulam stability of the functional equation (2.1) in random normed space.

Theorem 3.1. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\varphi : X^2 \to Z$ be a function such that there exists $0 < \alpha < \frac{1}{27}$ such that*

$$\mu'_{\varphi(\frac{x}{3},\frac{y}{3})}(t) \geq \mu'_{\alpha\varphi(x,y)}(t) \tag{3.1}$$

for all $x, y \in X$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{27^n\varphi(\frac{x}{3^n},\frac{y}{3^n})}(t) = 1$ for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping with $f(0) = 0$ and such that

$$\mu_{3f(x+3y)+f(3x-y)-15f(x+y)-15f(x-y)-80f(y)}(t) \geq \mu'_{\varphi(x,y)}(t) \tag{3.2}$$

for all $x, y \in X$ and $t > 0$, then the limit $C(x) = \lim_{n\to\infty} 27^n f\left(\frac{x}{3^n}\right)$ exist for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\frac{\alpha\varphi(x,0)}{1-27\alpha}}(t). \tag{3.3}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (3.2), we see that

$$\mu_{f(x)-27f(\frac{x}{3})}(t) \geq \mu'_{\varphi\left(\frac{x}{3},0\right)}(t) \tag{3.4}$$

for all $x \in X$. Replacing x by $\frac{x}{3^n}$ in (3.4) and using (3.1), we obtain

$$\mu_{27^{n+1}f\left(\frac{x}{3^{n+1}}\right)-27^n f\left(\frac{x}{3^n}\right)}(t) \geq \mu'_{3^n\varphi\left(\frac{x}{3^{n+1}},0\right)}(t) \geq \mu'_{27^n\alpha^{n+1}\varphi(x,0)}(t).$$

On the other hand

$$\begin{aligned} \mu_{27^n f\left(\frac{x}{3^n}\right)-f(x)}(t) &= \mu_{\frac{\sum_{k=0}^{n-1} 27^{k+1} f(\frac{x}{3^{k+1}})-27^k f(\frac{x}{3^k})}{\sum_{k=0}^{n-1} 27^k\alpha^{k+1}}}(t) \\ &\geq T_{k=0}^{n-1}\left(\mu'_{\varphi(x,0)}(t)\right) = \mu'_{\varphi(x,0)}(t). \end{aligned} \tag{3.5}$$

This implies that

$$\mu_{27^n f\left(\frac{x}{3^n}\right)-f(x)}(t) \geq \mu'_{\sum_{k=0}^{n-1} 27^k\alpha^{k+1}\varphi(x,0)}(t). \tag{3.6}$$

Replacing x by $\frac{x}{27^p}$ in (3.6), we obtain

$$\mu_{27^{n+p} f\left(\frac{x}{3^{n+p}}\right)-27^p f\left(\frac{x}{3^p}\right)}(t) \geq \mu'_{\sum_{k=p}^{n+p-1} 27^k\alpha^{k+1}\varphi(x,0)}(t). \tag{3.7}$$

Since $\lim_{p,n\to\infty}\mu'_{\sum_{k=p}^{n+p-1}27^k\alpha^{k+1}\varphi(x,0)}(t) = 1$, it follows that $\left\{27^n f(\frac{x}{3^n})\right\}$ is a Cauchy sequence in a complete RN-space $(Y,\mu,\min)$ and so there exists a point $C(x)\in Y$ such that

$$\lim_{n\to\infty}27^n f\left(\frac{x}{3^n}\right) = C(x).$$

It follows for (3.6) that , for any $\epsilon>0$,

$$\begin{aligned}(3.8)\quad \mu_{A(x)-f(x)}(t+\epsilon) &\geq T\left(\mu_{C(x)-27^n f(\frac{x}{3^n})}(\epsilon), \mu_{27^n f(\frac{x}{3^n})-f(x)}(t)\right)\\ &\geq T\left(\mu_{C(x)-27^n f(\frac{x}{3^n})}(\epsilon), \mu'_{\sum_{k=0}^{n-1}27^k\alpha^{k+1}\varphi(x,0)}(t)\right).\end{aligned}$$

Taking $n\to\infty$ in (3.8), we get

$$\mu_{C(x)-f(x)}(t+\epsilon) \geq \mu'_{\frac{\alpha\varphi(x,0)}{1-27\alpha}}(t). \quad (3.9)$$

Since ϵ is arbitrary, by taking $\epsilon\to 0$ in (3.9), we get

$$\mu_{C(x)-f(x)}(t) \geq \mu'_{\frac{\alpha\varphi(x,0)}{1-27\alpha}}(t).$$

Replacing x and y by $\frac{x}{3^n}$ and $\frac{y}{3^n}$ in (3.2), respectively, we get

$$\mu_{3.27^n f\left(\frac{x+3y}{3^n}\right)+27^n f\left(\frac{x-3y}{3^n}\right)-15.27^n f\left(\frac{x+y}{3^n}\right)-15.27^n f\left(\frac{x-y}{3^n}\right)-80.27^n f\left(\frac{y}{3^n}\right)}(t) \geq \mu'_{27^n\varphi\left(\frac{x}{3^n},\frac{y}{3^n}\right)}(t)$$

for all $x,y\in X$ and $t>0$. Since $\lim_{n\to\infty}\mu'_{27^n\varphi\left(\frac{x}{3^n},\frac{y}{3^n}\right)}(t)=1$, we conclude that A satisfies (2.1).

On the other hand

$$27C\left(\frac{x}{3}\right)-C(x) = \lim_{n\to\infty}27^{n+1}f\left(\frac{x}{3^{n+1}}\right)-\lim_{n\to\infty}27^n f\left(\frac{x}{3^n}\right)=0.$$

Therefore, the mapping $C: X\to Y$ is cubic. To prove the uniqueness of the additive mapping C, assume that there exists another additive mapping $L: X\to Y$ which satisfies (3.3). Then we have

$$\begin{aligned}\mu_{C(x)-L(x)}(t) = \lim_{n\to\infty}\mu_{27^n C\left(\frac{x}{3^n}\right)-27^n L\left(\frac{x}{3^n}\right)}(t) &\geq \lim_{n\to\infty}\mu'_{\varphi\left(\frac{x}{3^n},0\right)}\left(\frac{(1-27\alpha)t}{3^{n+1}\alpha}\right)\\ &\geq \lim_{n\to\infty}\mu'_{\varphi(x,0)}\left(\frac{(1-27\alpha)t}{3^{n+1}\alpha^{n+1}}\right).\end{aligned}$$

Since $\lim_{n\to\infty}\lim_{n\to\infty}\mu'_{\varphi(x,0)}\left(\frac{(1-27\alpha)t}{3^{n+1}\alpha^{n+1}}\right)=1$. Therefore, it follows that $\mu_{C(x)-D(x)}(t)=1$ for all $t>0$ and so $C(x)=L(x)$. This completes the proof. □

Corollary 3.1. *Let X be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $p > 1$ and $z_0 \in Z$. If $f : X \to Y$ be a mapping with $f(0) = 0$ and such that*

$$\mu_{3f(x+3y)+f(3x-y)-15f(x+y)-15f(x-y)-80f(y)}(t) \geq \mu'_{(\|x\|^p+\|y\|^p)z_0}(t)$$

for all $x, y \in X$ and $t > 0$, then the limit $C(x) = \lim_{n\to\infty} 27^n f\left(\frac{x}{3^n}\right)$ exist for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\frac{\|x\|^p z_0}{27^p-27}}(t)$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 27^{-p}$ and $\varphi : X^2 \to Z$ be a mapping defined by $\varphi(x, y) = (\|x\|^p + \|y\|^p)z_0$. Then, from Theorem 3.1, the conclusion follows. □

Acknowledgement

Dong Yun Shin(corresponding author) was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

References

[1] H. Azadi Kenary, *Hyres-Rassias stability of the Pexiderial functional equation,* to appear in Ital. J. Pure Appl. Math.

[2] H. Azadi Kenary, *Stability of a Pexiderial functional equation in random normed spaces,* Rend. Circ. Mat. Palermo, DOI: 10.1007/s12215-011-0027-5.

[3] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequat. Math. 27(1984), 76-86.

[4] S. Czerwik, *Functional Equations and Inequalities in several variables*, World Scientific, River Edge, NJ, 2002.

[5] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994), 431-436.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA 27 (1941), 222-224.

[7] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[8] R. Saadati, M. Vaezpour and Y. J. Cho, *A note to paper "On the stability of cubic mappings and quartic mappings in random normed spaces"*, J. Ineq. Appl., Volume 2009, Article ID 214530, doi: 10.1155/2009/214530.

[9] B. Schewizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland, New York, USA, 1983.

[10] F. Skof, *Local properties and approximation of operators*, Rend. Sem. Mat. Fis. Milano 53 (1983), 113-129.

[11] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons, 1964.

Received: October, 2011