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Fuzzy Estimators in Expert Systems

D. S. Sfiris and B. K. Papadopoulos

Section of Mathematics, Department of Civil Engineering
Democritus University of Thrace
12 Vasilissis Sofias Str., 67100 Xanthi, Greece
dsfiris@civil.duth.gr

Abstract

In this paper we consider the use of fuzzy estimators –a new and promising approach of estimating the parameters of a probability distribution from statistical samples– to represent mathematical knowledge in expert systems. A class of fuzzy estimators suitable for fuzzy arithmetics generalizes the existing approaches and it is used to derive the fuzzy estimators for the parameters of the normal distribution. The fuzzy binary operations of addition, subtraction, multiplication and division are defined, their explicit and unique membership functions are constructed, and a ranking method to deal with fuzzy comparisons is proposed. Finally, a Prolog-based implementation for performing fuzzy computational tasks is discussed.

Keywords: arithmetic operations; fuzzy estimators; knowledge representation; statistical methods

1. Introduction

Human brain has an inherent ability to cope with uncertainty. One of the early successful expert systems to deal with uncertainty in data and reasoning was the famous MYCIN system (Buchanan and Shortliffe 1984). Its essential idea was the introduction of certainty factors, as ad-hoc measures of belief attached to rules and facts, to express partial knowledge and to produce the final predictions based on such rough knowledge bases.

Since the mid-eighties, a shift from probability theory to fuzzy logic for the management of uncertainty in expert systems became apparent and fuzzy expert systems were developed to capture in a more natural way the reasoning process of human experts (Zadeh 1983). As a result, fuzzy expert system languages such as the FLOPS (Siler 1987), FuzzyProlog (Hinde 1986), FuzzyCLIPS (Orchard 2004) and others, became more expressive and capable than their classical counterparts (OPS, Prolog, CLIPS) by extending their reasoning mechanisms to deal with fuzzy constructs such as linguistic IF…THEN rules, fuzzy facts or other fuzzy statements functioning as rules and facts (i.e. fuzzy Horn clauses in Prolog) (Martin *et al*. 1987, Leung and Lam 1989, Tsutomu *et al*. 1993, Pan *et al*. 1998). These features have made expert systems a real success in all those application domains where knowledge involved is mainly declarative (Gallanti *et al.* 1985).

However, it has been reported in the literature that in addition to capturing knowledge in linguistic form, real-world problems dictate the need to include mathematical knowledge as well (Wagman *et al.* 1994), namely arithmetic functions, arithmetic operations, comparisons on numbers and assignments of variables. Clearly this requires a procedural knowledge representation scheme to allow algorithmic knowledge to be uniformly integrated with heuristic declarative knowledge (Gallanti *et al.* 1985).

Although expert system languages, Prolog being a prime paradigm, have been extended to deal with mathematical knowledge on real numbers and/or real intervals, to our knowledge none has incorporated fuzzy estimators in expert systems. Our interest in this paper is to establish an efficient method, capable to represent fuzzy mathematical knowledge in the particular case were statistical data are available. We are motivated by two research studies: (a) FLOPS (Buckley, Siler and Tucker 1986), a production fuzzy expert system which provided support for logical relations between fuzzy numbers and suggested the use of fuzzy operations, and (b) the work of Wagman, Schneider and Schnaider (1994) were a method for incorporating arithmetic expressions in fuzzy expert systems was developed using probability theory to evaluate these arithmetic expressions.

Instead, in this work, we investigate the use of fuzzy estimators –a new approach of estimating the parameters of a probability distribution from statistical samples (Buckley 2005)–. In fuzzy expert systems, this approach allows us to (a) construct fuzzy numbers based on statistical samples rather than expert evaluations, (b) perform fuzzy arithmetic operations efficiently using well defined formulae containing solely the statistical parameters taken from samples and (c) evaluate arithmetic comparisons using a fuzzy formula derived from the above mentioned fuzzy estimators. The proposed framework provides the necessary mathematical infrastructure required for such an implementation.

The rest of the paper is organized as follows: Section 2 introduces the basic notions and definitions from the theory of fuzzy numbers which will be used herein. In Section 3, a generalization of the current fuzzy estimation methods is developed in order to be practicable for fuzzy arithmetics and to give the reader the ability to construct fuzzy numbers of desired confidence from statistical data. In Section 4, we concentrate on fuzzy estimators for the parameters of the normal distribution. In Section 5, the binary fuzzy arithmetic operations of addition, subtraction, multiplication and division are defined and their membership functions are constructed. In Section 6, a fuzzy ranking method using the fuzzy estimators of the normal distribution is proposed to deal with fuzzy comparisons and in Section 7 an integration of fuzzy estimators into Prolog is discussed.

2. Fuzzy numbers and their operations

Let us introduce the mathematical notation, definitions and properties (Klir and Yuan 1995) that we will need throughout the paper.

Definition 2.1: Let X be a universal set. Every function of the form $\mathrm{A}:\mathrm{X}\rightarrow[0,1]$ is called a *fuzzy set* or a fuzzy subset of X, where $\mathrm{A}(x)$ is interpreted as the "membership degree" of x in the fuzzy set A.

Definition 2.2: The *α-cuts* of a fuzzy set A are the defined by the crisp sets ${}^{\alpha}\mathrm{A}=\{x\in\mathrm{R}:\mathrm{A}(x)\geq\alpha\}$. It known that the α-cuts determine the fuzzy set A.

Definition 2.3: A is a *fuzzy number* if the following conditions hold:

1) A is normal, that is there exists $\mathrm{x}\in\Re$ such that $\mathrm{A}(x)=1$.
2) A is a convex fuzzy set, that is for every $\mathrm{t}\in[0,1]$ and $\mathrm{x}_1,\mathrm{x}_2\in\Re$, we have
3) $\mathrm{A}((1-\mathrm{t})\mathrm{x}_1+\mathrm{t}\mathrm{x}_2)\geq\min\{\mathrm{A}(\mathrm{x}_1),\mathrm{A}(\mathrm{x}_2)\}$
4) A is upper semi-continuous,
5) The support of A $\mathrm{suppA}=\overline{\bigcup_{\alpha\in(0,1]}{}^{\alpha}\mathrm{A}}=\overline{\{\mathrm{x}:\mathrm{A}(\mathrm{x})>0\}}$ is compact.

Definition 2.4: If A is a fuzzy number then its α-cuts can be written as intervals of the form ${}^{\alpha}\mathrm{A}=[\mathrm{A}_L(\alpha),\mathrm{A}_R(\alpha)]$, where $\mathrm{A}_L(\alpha),\mathrm{A}_R(\alpha)$ can be regarded as functions on $(0,1]$. Then,

1) $\mathrm{A}_L(\alpha)$ is non-decreasing and left continuous,
2) $\mathrm{A}_R(\alpha)$ is non-increasing and left continuous,
3) $\mathrm{A}_L(\alpha)\leq\mathrm{A}_R(\alpha)$.

Definition 2.5: The *membership function* of a fuzzy number A can be expressed as

$$\mathrm{A}(x)=\begin{cases}0, & x<a_1\\ \mathrm{A}_L(x), & a_1\leq x\leq a_2\\ \mathrm{A}_R(x), & a_2\leq x\leq a_3\\ 0, & x>a_3\end{cases} \tag{1}$$

where $\mathrm{A}_L:[a_1,a_2]\rightarrow[0,1]$ and $\mathrm{A}_R:[a_3,a_4]\rightarrow[0,1]$ are left and right membership function of the fuzzy number A.

Zadeh's extension principle extended the classical operations between real numbers to their fuzzy counterparts (Klir and Yuan 1995).

Definition 2.6: Let A and B be two fuzzy numbers. Then according to the extension principle any crisp function $f:\mathrm{X}\rightarrow\mathrm{Y}$ induces a fuzzy valued function

$F : \mathrm{P}(\mathrm{X}) \to \mathrm{P}(\mathrm{Y})$ where $\mathrm{P}(.)$ denotes a fuzzy power set. Using this principle, a fuzzy binary arithmetic operation is defined as

$$C(z) = \max_{z=x\oplus y}\left[\mathrm{A}(x) \wedge \mathrm{B}(y)\right],\ (x, y, z) \in \Re^3$$

where $\oplus \in \{+, -, *, /\}$.

Remark 2.1: The above definition can be easily generalized to a family of sets, assuming that f is a mapping from a Cartesian product $\mathrm{X}_1 \times \mathrm{X}_2 \times \cdots \times \mathrm{X}_n$ into Y.

Under the extension principle, arithmetic operations require rather tedious sequences of convolution-like operations on the support elements of fuzzy numbers and logical operations (Filev and Yager 1997). A more efficient and well accepted method is to consider arithmetic operations between fuzzy numbers as arithmetic operations on basis of closed bounded intervals on real numbers by employing their α-cuts representation (Dubois and Prade 1978).

Definition 2.7: Let A and B be two fuzzy numbers and ${}^{\alpha}\mathrm{A} = \left[\mathrm{A}_L(\alpha), \mathrm{A}_R(\alpha)\right]$, ${}^{\alpha}\mathrm{B} = \left[\mathrm{B}_L(\alpha), \mathrm{B}_R(\alpha)\right]$ be α-cuts, $\alpha \in (0,1]$, of A and B respectively. The α-cut operations of the fuzzy arithmetic addition $(+)$, subtraction $(-)$, multiplication $(*)$ and division $(/)$ are:

1) ${}^{\alpha}[\mathrm{A}(+)\mathrm{B}] = \left[\mathrm{A}_L^{\alpha} + \mathrm{B}_L^{\alpha}, \mathrm{A}_R^{\alpha} + \mathrm{B}_R^{\alpha}\right]$

2) ${}^{\alpha}[\mathrm{A}(-)\mathrm{B}] = \left[\mathrm{A}_L^{\alpha} - \mathrm{B}_R^{\alpha}, \mathrm{A}_R^{\alpha} - \mathrm{B}_L^{\alpha}\right]$

3) ${}^{\alpha}[\mathrm{A}(*)\mathrm{B}] = \left[\min\left(\mathrm{A}_L^{\alpha} \cdot \mathrm{B}_L^{\alpha}, \mathrm{A}_L^{\alpha} \cdot \mathrm{B}_R^{\alpha}, \mathrm{A}_R^{\alpha} \cdot \mathrm{B}_L^{\alpha}, \mathrm{A}_R^{\alpha} \cdot \mathrm{B}_R^{\alpha}\right), \max\left(\mathrm{A}_L^{\alpha} \cdot \mathrm{B}_L^{\alpha}, \mathrm{A}_L^{\alpha} \cdot \mathrm{B}_R^{\alpha}, \mathrm{A}_R^{\alpha} \cdot \mathrm{B}_L^{\alpha}, \mathrm{A}_R^{\alpha} \cdot \mathrm{B}_R^{\alpha}\right)\right]$

4) ${}^{\alpha}[\mathrm{A}(/)\mathrm{B}] = \left[\min\left(\mathrm{A}_L^{\alpha} / \mathrm{B}_L^{\alpha}, \mathrm{A}_L^{\alpha} / \mathrm{B}_R^{\alpha}, \mathrm{A}_R^{\alpha} / \mathrm{B}_L^{\alpha}, \mathrm{A}_R^{\alpha} / \mathrm{B}_R^{\alpha}\right), \max\left(\mathrm{A}_L^{\alpha} / \mathrm{B}_L^{\alpha}, \mathrm{A}_L^{\alpha} / \mathrm{B}_R^{\alpha}, \mathrm{A}_R^{\alpha} / \mathrm{B}_L^{\alpha}, \mathrm{A}_R^{\alpha} / \mathrm{B}_R^{\alpha}\right)\right]$

where $\mathrm{A}_L^{\alpha} = \mathrm{A}_L(\alpha), \mathrm{A}_R^{\alpha} = \mathrm{A}_R(\alpha), \mathrm{B}_L^{\alpha} = \mathrm{B}_L(\alpha), \mathrm{B}_R^{\alpha} = \mathrm{B}_R(\alpha)$ for short and $0 \notin {}^{\alpha}\mathrm{B}$ for the division.

It is known that when operating with fuzzy numbers, the result of calculations depends on the type of the membership functions of these numbers. Various types have been proposed in the literature (Kreinovich, Quintana and Reznik 1992, Fodor and Bede 2006). In practice, a suitable membership function is often assumed a-priori and its parameters are determined either (a) by domain experts or (b) from training data using a machine learning technique (Hong and Lee 1996). Although the first approach is often used, it may not be accurate enough in problems where expert knowledge is not available. In this paper we focus on the second approach from a point of view of statistics and fuzzy set theory.

3. Fuzzy estimators based on confidence intervals

Fuzzy estimation is a method to construct fuzzy number estimators for the parameters of probability density (mass) functions using statistical samples (Buckley 2005). He gave a construction procedure which uses the set of confidence intervals to express the uncertainties of a parameter producing a triangular shaped fuzzy number.

In this section we present a generalization by introducing an appropriate transformation on the confidence intervals, without changing the triangular shape of the curve. There are two reasons for doing this: (a) to ensure that the base of the produced fuzzy estimators is closed and bounded, thus making them suitable for fuzzy arithmetics, and (b) to give the reader the ability to choose a desired parameter $\gamma \in [0,1)$ of $(1-\gamma)100\%$ confidence interval for θ as the base of a fuzzy estimator. Hence, the proposed approach subsumes the existing method of Buckley (2005) (by considering it as a special case when $\gamma = 0$).

First, we prove in the following theorem the aforementioned generalization and then we describe the construction procedure.

Theorem 3.1: Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with unknown parameter θ and $[\theta_1(\beta), \theta_2(\beta)]$ denotes the $(1-\beta)100\%$ confidence intervals for θ. A triangular shaped fuzzy number $\overline{\theta}$ can be constructed, call it the compact fuzzy estimator of θ, from the following intervals:

$$^{2g(\beta)}\overline{\theta} = [\theta_1(2g(\beta)), \theta_2(2g(\beta))] \text{ , } \beta \in [0,1] \tag{2}$$

which are exactly the $(1-2g(\beta))100\%$ confidence intervals for θ, obtained by applying an appropriate transformation $g(\beta)$ on the confidence intervals $(1-\beta)100\%$of θ, where $g(\beta)$ is a monotonic increasing, continuous and onto function

$$g(\beta):[0,1] \to \left[\frac{\gamma}{2}, 0.5\right], \ \gamma \in [0,1). \tag{3}$$

Proof. Let $p(\beta) = \theta_1(2g(\beta))$ and $q(\beta) = \theta_2(2g(\beta))$ be functions on $(0,1]$.
From Definition 2.4, $p(\beta)$ and $q(\beta)$ must satisfy the following conditions:

(i) $p(\beta) \le q(\beta)$,

(ii) $p(\beta)$ is non-decreasing and left continuous and

(iii) $q(\beta)$ is non-increasing and left continuous.

Since the $(1-2g(\beta))100\%$ confidence intervals are nested, it holds that $\theta_1(2g(\beta)) \le \theta_2(2g(\beta))$. To prove (ii) and (iii), we consider initially that $\gamma = 0$. Then (3) becomes $g(\beta):[0,1] \to [0, 0.5]$ and (2) is reduced to $[\theta_1(\beta), \theta_2(\beta)]$, but as Buckley (2005) has shown, a triangular shaped fuzzy number can be constructed from $[\theta_1(\beta), \theta_2(\beta)]$. Therefore $\theta_1(\beta)$ is non-decreasing and $\theta_1(\beta)$ is non-increasing. Hence, it holds that $p(\beta)$ is non-decreasing as composition of two non-decreasing functions, and $p(\beta)$ is non-increasing as a composition of a

non-increasing with a non-decreasing function. Notice that $p(\beta)$ and $q(\beta)$ are continuous as a composition of two continuous functions. Hence, according to Theorem 2.1 of Wu and Zhang (2000), there exists a unique fuzzy set that can be constructed from (2). To prove that this fuzzy set is a fuzzy number, it is sufficient to prove that all the α-cuts are contained in a closed and bounded interval. Obviously, this interval is the
$^{2g(0)}\overline{\theta} = \left[\theta_1\left(2g(0)\right), \theta_2\left(2g(0)\right)\right] = \left[\theta_1(\gamma), \theta_2(\gamma)\right]$
which is exactly the $(1-\gamma)100\%$ confidence interval for θ.

The generalized construction procedure goes as follows. Let X be a random variable with probability density function $f(x;\theta)$ for single parameter θ. Assume that θ is unknown and it must be estimated from a random sample $X_1,...,X_n$. Denote the $(1-\beta)100\%$ confidence intervals for θ by $\left[\theta_1(\beta), \theta_2(\beta)\right]$ for all $0\leq \beta \leq 1$. Select a parameter $\gamma \in [0,1)$ of a $(1-\gamma)100\%$ confidence interval for θ as the base of a fuzzy estimator. Apply a monotonic increasing, continuous and onto transformation function $g(\beta):[0,1] \to \left[\frac{\gamma}{2}, 0.5\right]$ on the $(1-\beta)100\%$ confidence intervals to obtain $^{2g(\beta)}\overline{\theta} = \left[\theta_1\left(2g(\beta)\right), \theta_2\left(2g(\beta)\right)\right]$, $\beta \in [0,1]$.Then place these confidence intervals, one on top of the other, to produce a triangular shaped fuzzy number $\overline{\theta}$ whose α-cuts are the following confidence intervals: $^{\alpha}\overline{\theta} = \left[\theta_1(\alpha), \theta_2(\alpha)\right]$, $\alpha = 2g(\beta)$ for $0\leq \alpha \leq 1$.

Hereinafter when we refer to fuzzy estimators we will refer to the above mentioned general class of fuzzy estimators. In the case of $\gamma = 0$ a fuzzy estimator is asymptotic with α-cuts exactly the $(1-\beta)100\%$ confidence intervals for the estimated parameter. Hence, the fuzzy estimator extends from -∞ to +∞ and one has to drop the graph of the near the base to make a complete fuzzy number. Otherwise, when $\gamma \in [0,1)$, an appropriate function $g(\beta):[0,1] \to \left[\frac{\gamma}{2}, 0.5\right]$ is chosen to map the $(1-\beta)100\%$ confidence intervals. Thus a class of fuzzy estimators non-asymptotic in their base is obtained. A proper selection of $\gamma \neq 0$ and $g(\beta)$ is application dependant. For simplicity, we suggest the use of the linear transformation function $g(\beta) = \left(\frac{1}{2} - \frac{\gamma}{2}\right)\beta + \frac{\gamma}{2}$ for each $\gamma \in [0,1)$. Fig.1 illustrates both approaches.

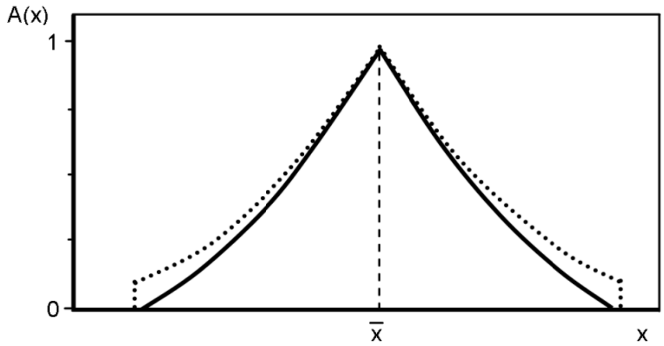

Figure 1. Comparison of two fuzzy estimators for the mean μ of the normal distribution. The dotted line depicts a fuzzy estimator constructed with $\gamma = 0$. The solid line depicts the same fuzzy estimator with $\gamma = 0.90$.

Recently, Falsafain *et al.* (2008) proved the derivation of explicit and unique membership functions of fuzzy estimators. Their theorem also holds for $\gamma = 0$. Here we extend it for every $\gamma \in [0,1)$.

Theorem 3.2: Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n from a distribution with unknown parameter θ. If, based on observations $x_1, x_2, ...x_n$, we consider $[\theta_1(\beta), \theta_2(\beta)]$ as a $(1-\beta)100\%$ confidence interval for θ, then a compact fuzzy estimator of θ is a fuzzy number with the following membership function:
$\overline{\theta}(u) = \min\{(p)^{-1}(u), (-q)^{-1}(-u), 1\}$, where $p = \theta_1 \circ 2g$, $q = \theta_2 \circ 2g$ denote compositions of functions and $g(\beta):[0,1] \to \left[\frac{\gamma}{2}, 0.5\right]$ is monotonic increasing, continues and onto function, where $\gamma \in [0,1)$.

Proof. It was shown in Theorem 3.1 that the α-cuts of the compact fuzzy estimator are ${}^{2g(\beta)}\overline{\theta} = [(\theta_1 \circ 2g)(\beta), (\theta_2 \circ 2g)(\beta)]$ or ${}^{\alpha}\overline{\theta} = [p(\beta), q(\beta)]$ where $\alpha = 2g(\beta)$. If $u \in {}^{\alpha}\overline{\theta}$, then $p(\beta) \le u \le q(\beta)$. From $p(\beta) \le u$ we obtain $\beta \le (p)^{-1}(u)$, and from $u \le q(\beta)$ we obtain $\beta \le (-q)^{-1}(-u)$. Moreover, $\beta \le 1$, therefore
$\beta = \min\{(p)^{-1}(u), (-q)^{-1}(-u), 1\} = \overline{\theta}(u)$.

4. Estimating the fuzzy mean and fuzzy standard deviation of the normal distribution from statistical samples

The above allow us to construct both the α-cuts and membership function of fuzzy estimators for the parameters of a given distribution. In this section, we concentrate on the fuzzy estimators for the normal distribution. We deal with the fuzzy mean and fuzzy standard deviation (small and large samples). Our approach is illustrated with examples.

Theorem 4.1: *(Fuzzy mean)* Let $X_1, X_2, ..., X_n$ be a random sample and let $x_1, x_2, ...x_n$ be sample values assumed by the sample. We consider $\left[\theta_1(\beta), \theta_2(\beta)\right]$ as the $(1-\beta)100\%$ confidence intervals for μ. Let also $\gamma \in [0,1)$ and Φ denotes the standard normal distribution function. If the sample size is large enough, then the compact fuzzy estimator for μ is a fuzzy number with the following membership function

$$\overline{\mu}(u) = \begin{cases} \dfrac{2-\gamma}{1-\gamma} - \dfrac{2}{1-\gamma}\Phi\left(\dfrac{|\overline{x}-u|}{\sigma/\sqrt{n}}\right) & \text{if } \overline{x} - \dfrac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1-\dfrac{\gamma}{2}\right) \le u \le \overline{x} + \dfrac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1-\dfrac{\gamma}{2}\right) \\ 0, \text{ otherwise} \end{cases} \quad (4)$$

where $g(\beta) = \left(\frac{1}{2} - \frac{\gamma}{2}\right)\beta + \frac{\gamma}{2} \quad \left(g : [0,1] \to \left[\frac{\gamma}{2}, 0.5\right]\right)$.

Proof: According to Theorem 3.1, the fuzzy estimator for μ is a fuzzy number the α-cuts of which are exactly the closed intervals

$$^{2g(\beta)}\overline{\mu} = \left[\overline{x} - z_{g(\beta)}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{g(\beta)}\frac{\sigma}{\sqrt{n}}\right], \beta \in [0,1]. \quad (5)$$

We now construct the membership function of this fuzzy number.

Let $p(\beta) = (\theta_1 \circ 2g)(\beta) = \overline{x} - z_{g(\beta)}\frac{\sigma}{\sqrt{n}}$ and $q(\beta) = (\theta_2 \circ 2g)(\beta) = \overline{x} + z_{g(\beta)}\frac{\sigma}{\sqrt{n}}$ be functions on $(0,1]$ where $\circ$ denotes composition of functions.

We find $p^{-1}(u)$ and $q^{-1}(u)$ as follows:

Let

$$\overline{x} - z_{g(\beta)}\frac{\sigma}{\sqrt{n}} = u \Leftrightarrow z_{g(\beta)} = \frac{\overline{x}-u}{\sigma/\sqrt{n}} \Leftrightarrow \Phi^{-1}\left(1-g(\beta)\right) = \frac{\overline{x}-u}{\sigma/\sqrt{n}} \Leftrightarrow$$

$$1-g(\beta) = \Phi\left(\frac{\overline{x}-u}{\sigma/\sqrt{n}}\right) \Leftrightarrow \left(\frac{1-\gamma}{2}\right)\beta + \frac{\gamma}{2} = 1 - \Phi\left(\frac{\overline{x}-u}{\sigma/\sqrt{n}}\right) \Leftrightarrow$$

$$\beta = \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma}\Phi\left(\frac{\overline{x}-u}{\sigma/\sqrt{n}}\right) = p^{-1}(u) \quad (6)$$

Since $0 \le \beta \le 1$

$$0 \le \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma}\Phi\left(\frac{\overline{x}-u}{\sigma/\sqrt{n}}\right) \le 1 \Leftrightarrow \frac{1}{2} \le \Phi\left(\frac{\overline{x}-u}{\sigma/\sqrt{n}}\right) \le \frac{2-\gamma}{2} \Leftrightarrow$$

$$\Phi^{-1}\left(\frac{1}{2}\right) \le \frac{\overline{x}-u}{\sigma/\sqrt{n}} \le \Phi^{-1}\left(1-\frac{\gamma}{2}\right) \Leftrightarrow 0 \le \overline{x}-u \le \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1-\frac{\gamma}{2}\right) \Leftrightarrow$$

$$\overline{x} - \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1-\frac{\gamma}{2}\right) \le u \le \overline{x} \tag{7}$$

Similarly for the other side of the α-cuts interval of (5), we find

$$q^{-1}(u) = \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma}\Phi\left(\frac{u-\overline{x}}{\sigma/\sqrt{n}}\right) \tag{8}$$

$$\overline{x} \le u \le \overline{x} + \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1-\frac{\gamma}{2}\right) \tag{9}$$

By combining (6),(7),(8),(9) and based on Theorem 3.2 we obtain

$$\overline{\mu}(u) = \begin{cases} \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma}\Phi\left(\frac{|\overline{x}-u|}{\sigma/\sqrt{n}}\right) & if\ \overline{x} - \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1-\frac{\gamma}{2}\right) \le u \le \overline{x} + \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1-\frac{\gamma}{2}\right) \\ 0, \text{ otherwise} \end{cases}$$

Thus, the theorem has been proved.

Numerical example Suppose $n = 100$, $\sigma^2 = 16$ and $\overline{x} = 40$ from a normally distributed random sample with unknown mean μ. To construct a compact fuzzy estimator for μ, we begin with a value for γ. For our example let $\gamma = 0.1$ which corresponds to the 90% confidence interval for μ. According to Theorem 4.1 its membership function is given by

$$\overline{\mu}(u) = \begin{cases} \frac{2-0.1}{1-0.1} - \frac{2}{1-0.1}\Phi\left(\frac{|40-u|}{4/\sqrt{100}}\right) & if\ \ u_1 \le u \le u_2 \\ 0, \text{ otherwise} \end{cases}$$

$$u_1 = 40 - \frac{4}{\sqrt{100}}\Phi^{-1}\left(1-\frac{0.1}{2}\right)$$

$$u_2 = 40 + \frac{4}{\sqrt{100}}\Phi^{-1}\left(1-\frac{0.1}{2}\right)$$

or

$$\overline{\mu}(u) = \begin{cases} 2.111 - 2.222\Phi(|100-2.5u|) & if\ \ 39.34 \le u \le 40.66 \\ 0, \text{ otherwise} \end{cases}$$

The graph of this estimator is depicted in Fig.2.

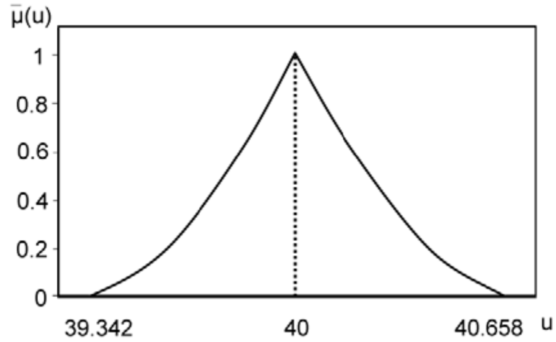

Figure 2: Compact fuzzy estimator for μ.

Remark 4.1: If the sample size is small and it has been drawn from a normally distributed population with unknown σ, then σ in formula (5) is replaced by s and $z_{g(\beta)}$ is replaced by $t_{g(\beta)}$, where $t_{g(\beta)} = T^{-1}(1-g(\beta))$ and T denotes the t-student distribution with $\nu = n-1$ degrees of freedom.

Theorem 4.2: *(Fuzzy standard deviation – small sample)* Let $X_1, X_2, ..., X_n$ be a random sample and let $x_1, x_2, ...x_n$ be sample values assumed by the sample. We consider $[\theta_1(\beta), \theta_2(\beta)]$ as the $(1-\beta)100\%$ confidence intervals for σ^2. Let also $\gamma \in [0,1)$, F denotes the distribution function of chi-squared with n-1 degrees of freedom and M is the median of the distribution. If the sample size is small, then the compact fuzzy estimator for σ^2 is a fuzzy number with the following membership function

$$\overline{\sigma}^2(u) = \begin{cases} \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma} F\left(\frac{(n-1)s^2}{u}\right) & if \quad \frac{(n-1)s^2}{F^{-1}\left(\frac{\gamma}{2}\right)} \le u \le \frac{(n-1)s^2}{M} \\ \frac{2}{1-\gamma} F\left(\frac{(n-1)s^2}{u}\right) - \frac{\gamma}{1-\gamma} & if \quad \frac{(n-1)s^2}{M} \le u \le \frac{(n-1)s^2}{F^{-1}\left(\frac{2-\gamma}{2}\right)} \\ 0, \text{ otherwise} & \end{cases} \tag{10}$$

where $g(\beta) = \left(\frac{1}{2} - \frac{\gamma}{2}\right)\beta + \frac{\gamma}{2} \quad \left(g:[0,1] \to \left[\frac{\gamma}{2}, 0.5\right]\right)$.

Proof: According to Theorem 3.1, the fuzzy estimator for σ^2 is a fuzzy number the α-cuts of which are exactly the closed intervals

$$^{2g(\beta)}\overline{\sigma}^2 = \left[\frac{(n-1)s^2}{\chi^2_{g(\beta)}}, \frac{(n-1)s^2}{\chi^2_{1-g(\beta)}}\right], \beta \in [0,1]. \tag{11}$$

Based on Theorem 3.2, the proof of the membership function is similar to the proof of Theorem 4.1.

Numerical example Assume we have a random sample $X_1, X_2, ... X_n$ of size $n = 10$ from a normally distributed population with unknown σ^2 and let $x_1, x_2, ... x_n$ be sample observations with sample variance $s^2 = 4$. If based on Theorem 4.2, we consider a $(1-\gamma)100\% = 90\%$ confidence interval for σ^2, then the compact fuzzy estimator $\overline{\sigma}^2$ is given by

$$\overline{\sigma}^2(u) = \begin{cases} 2.111 - 2.222F\left(\frac{36}{u}\right) & if \ \frac{36}{F^{-1}(0.95)} \le u \le \frac{36}{\text{M}} \\ 2.222F\left(\frac{36}{u}\right) - 0.111 & if \ \frac{36}{\text{M}} \le u \le \frac{36}{F^{-1}(0.05)} \\ 0, \text{ otherwise} \end{cases}$$

Since $F^{-1}(0.95) = 3.33$, $F^{-1}(0.05) = 16.925$ and $F^{-1}(0.5) = 8.35 = M$, we have

$$\overline{\sigma}^2(u) = \begin{cases} 2.111 - 2.222F\left(\frac{36}{u}\right) & if \ 2.127 \le u \le 4.311 \\ 2.222F\left(\frac{36}{u}\right) - 0.111 & if \ 4.311 \le u \le 10.811 \\ 0, \text{ otherwise} \end{cases}$$

Therefore, we have estimated the mean σ^2 to be a compact fuzzy number estimator almost 4.311. The support of the produced fuzzy number is the closed interval $[2.127, 10.811]$. The graph of this estimator is depicted in Fig. 3.

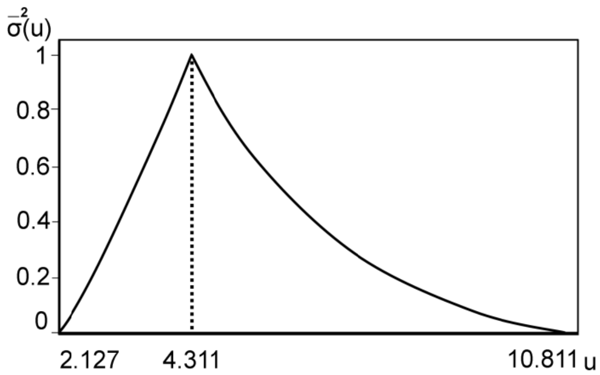

Figure 3: Compact fuzzy estimator for σ^2.

Theoreom 4.3: *(Fuzzy standard deviation – large sample)* Let $X_1, X_2, ..., X_n$ be a normally distributed random sample and let $x_1, x_2, ... x_n$ be sample values assumed by the sample. Let also $\gamma \in [0,1)$ and Φ denotes the standard normal distribution function. We consider $[\theta_1(\beta), \theta_2(\beta)]$ as the $(1-\beta)100\%$confidence intervals for

σ^2. If the sample size is large enough, then the compact fuzzy estimator for σ^2 is a fuzzy number with the following membership function

$$\overline{\sigma}^2(u)=\begin{cases}\frac{2-\gamma}{1-\gamma}-\frac{2}{1-\gamma}\Phi\left(\sqrt{\frac{n-1}{2}}\left(\frac{s^2}{u}-1\right)\right) & if \quad \frac{s^2}{1+\Phi^{-1}\left(1-\frac{\gamma}{2}\right)\sqrt{\frac{2}{n-1}}}\le u\le s^2\\ \frac{2-\gamma}{1-\gamma}-\frac{2}{1-\gamma}\Phi\left(\sqrt{\frac{n-1}{2}}\left(1-\frac{s^2}{u}\right)\right) & if \quad s^2\le u\le\frac{s^2}{1-\Phi^{-1}\left(1-\frac{\gamma}{2}\right)\sqrt{\frac{2}{n-1}}}\\ 0,\ \text{otherwise} & \end{cases} \tag{12}$$

where $g(\beta)=\left(\frac{1}{2}-\frac{\gamma}{2}\right)\beta+\frac{\gamma}{2}$ $\left(g:[0,1]\to\left[\frac{\gamma}{2},0.5\right]\right)$.

Proof: According to Theorem 3.1, the fuzzy estimator for σ^2 is a fuzzy number the α-cuts of which are exactly the closed intervals

$$^{2g(\beta)}\overline{\sigma}^2=\left[\frac{s^2}{1+z_{g(\beta)}\sqrt{\frac{2}{n-1}}},\frac{s^2}{1-z_{g(\beta)}\sqrt{\frac{2}{n-1}}}\right],\beta\in[0,1]. \tag{13}$$

Based on Theorem 3.2, the proof of the membership function is again similar to the proof of Theorem 4.1.

5. Fuzzy arithmetics using fuzzy estimators

In this section we present our methodology on fuzzy arithmetic operations using fuzzy estimators. We show that the operations can be defined directly via well-defined membership functions without resorting to α-cuts.

5.1 Fuzzy addition

We consider two fuzzy numbers M_1 and M_2 estimated from two normally distributed samples $X_1,X_2,...,X_n$ and $Y_1,Y_2,...,Y_n$ respectively. In order to realize the operation we need to find their α-cuts. These are, according to Definition 2.4,

$$^{\alpha}\mathrm{M}_1=\left[\underline{\mathrm{M}}_{1,\alpha},\overline{\mathrm{M}}_{1,\alpha}\right],\ ^{\alpha}\mathrm{M}_2=\left[\underline{\mathrm{M}}_{2,\alpha},\overline{\mathrm{M}}_{2,\alpha}\right]$$

Let us consider the fuzzy mean estimators of Theorem 4.1 obtained from the above samples. Then we have

$$^{2g(\beta)}\mathrm{M}_1=\left[\overline{t}_1-z_{g(\beta)}\frac{s_1}{\sqrt{n_1}},\overline{t}_1+z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right],\ ^{2g(\beta)}\mathrm{M}_2=\left[\overline{t}_2-z_{g(\beta)}\frac{s_2}{\sqrt{n_2}},\overline{t}_2+z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right],\beta\in[0,1]$$

were $\bar{t}_1, s_1, n_1$ and $\bar{t}_2, s_2, n_2$ are the sample means, sample standard deviations and sample sizes respectively. The fuzzy addition of M_1 and M_2 denoted by $\mathrm{M} = \mathrm{M}_1(+)\mathrm{M}_2$ is according to Definition 2.7

$$^{\alpha}\mathrm{M} = {}^{\alpha}\left[\mathrm{M}_1(+)\mathrm{M}_2\right] = \left[\underline{\mathrm{M}}_{1,\alpha} + \underline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{1,\alpha} + \overline{\mathrm{M}}_{2,\alpha}\right], \alpha \in (0,1].$$

Therefore, the α-cuts of the fuzzy addition are

$$^{2g(\beta)}\mathrm{M} = \left[\bar{t}_1 + \bar{t}_2 - z_{g(\beta)}\left(\frac{s_1}{\sqrt{n_1}} + \frac{s_2}{\sqrt{n_2}}\right), \bar{t}_1 + \bar{t}_2 + z_{g(\beta)}\left(\frac{s_1}{\sqrt{n_1}} + \frac{s_2}{\sqrt{n_2}}\right)\right], \beta \in [0,1]. \tag{14}$$

Moreover, to find the membership function of this operation, beginning with the left side of the α-cuts interval (14) let $u = \bar{t}_1 + \bar{t}_2 - z_{g(a)}\left(\frac{s_1}{\sqrt{n_1}} + \frac{s_2}{\sqrt{n_2}}\right)$, then

$$z_{g(\beta)} = \left(\frac{\bar{t}_1 + \bar{t}_2 - u}{\left(\frac{s_1}{\sqrt{n_1}} + \frac{s_2}{\sqrt{n_2}}\right)}\right) \Leftrightarrow \Phi^{-1}\left(1 - g(\beta)\right) = \left(\frac{\bar{t}_1 + \bar{t}_2 - u}{\left(\frac{\sqrt{n_2}s_1 + \sqrt{n_1}s_2}{\sqrt{n_1 n_2}}\right)}\right). \tag{15}$$

Since $g(\beta) = \left(\frac{1}{2} - \frac{\gamma}{2}\right)\beta + \frac{\gamma}{2}$ (as in Section 4) we obtain

$$\beta = \frac{2}{1-\gamma}\Phi\left(\frac{u - (\bar{t}_1 + \bar{t}_2)}{\left(\frac{\sqrt{n_2}s_1 + \sqrt{n_1}s_2}{\sqrt{n_1 n_2}}\right)}\right) + \frac{2-\gamma}{1-\gamma}, \gamma \in [0,1). \tag{16}$$

Finally, since $0 \le \beta \le 1$, (16) is true when

$$(\bar{t}_1 + \bar{t}_2) - \left(\frac{\sqrt{n_2}s_1 + \sqrt{n_1}s_2}{\sqrt{n_1 n_2}}\right)\Phi^{-1}\left(1 - \frac{\gamma}{2}\right) \le u \le (\bar{t}_1 + \bar{t}_2). \tag{17}$$

Similarly, for the right side of the α-cuts interval (14), let

$u = \bar{t}_1 + \bar{t}_2 + z_{g(\beta)}\left(\frac{s_1}{\sqrt{n_1}} + \frac{s_2}{\sqrt{n_2}}\right)$ to produce the relevant formulae.

From (16), (17) of the left side, the corresponding formulae for the right side and based on Theorem 3.1 we obtain the following fuzzy addition

$$\mathrm{M}(u) = \begin{cases} \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma}\Phi\left(\frac{(\bar{t}_1 + \bar{t}_2) - u}{\left(\frac{\sqrt{n_2}s_1 + \sqrt{n_1}s_2}{\sqrt{n_1 n_2}}\right)}\right) & \text{if } (\bar{t}_1 + \bar{t}_2) - \left(\frac{\sqrt{n_2}s_1 + \sqrt{n_1}s_2}{\sqrt{n_1 n_2}}\right)\Phi^{-1}\left(1 - \frac{\gamma}{2}\right) \le u \le (\bar{t}_1 + \bar{t}_2) \\ \frac{2-\gamma}{1-\gamma} - \frac{2}{1-\gamma}\Phi\left(\frac{u - (\bar{t}_1 + \bar{t}_2)}{\left(\frac{\sqrt{n_2}s_1 + \sqrt{n_1}s_2}{\sqrt{n_1 n_2}}\right)}\right) & \text{if } (\bar{t}_1 + \bar{t}_2) \le u \le (\bar{t}_1 + \bar{t}_2) + \left(\frac{\sqrt{n_2}s_1 + \sqrt{n_1}s_2}{\sqrt{n_1 n_2}}\right)\Phi^{-1}\left(1 - \frac{\gamma}{2}\right) \end{cases} \tag{18}$$

where $\gamma \in [0,1)$.

5.2 Fuzzy subtraction

The fuzzy subtraction of M_1 and M_2 is implemented as a fuzzy addition by changing the sign. Thus $\mathrm{M} = \mathrm{M}_1(-)\mathrm{M}_2$ is defined as

$$^{\alpha}\mathrm{M} = \left[\underline{\mathrm{M}}_{1,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\right] - \left[\underline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{2,\alpha}\right] = \left[\underline{\mathrm{M}}_{1,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\right] + \left[-\underline{\mathrm{M}}_{2,\alpha}, -\overline{\mathrm{M}}_{2,\alpha}\right], \alpha \in (0,1].$$

By Theorem 4.1, the α-cuts of the fuzzy subtraction are

$$^{2g(\beta)}\mathrm{M} = \left[\overline{t}_1 - \overline{t}_2 - z_{g(\beta)}\left(\frac{s_1}{\sqrt{n_1}} - \frac{s_2}{\sqrt{n_2}}\right), \overline{t}_1 - \overline{t}_2 + z_{g(\beta)}\left(\frac{s_1}{\sqrt{n_1}} - \frac{s_2}{\sqrt{n_2}}\right)\right], \beta \in [0,1] \quad (19)$$

and the membership function of the fuzzy subtraction turns out to be

$$\mathrm{M}(u) = \begin{cases} \dfrac{2-\gamma}{1-\gamma} - \dfrac{2}{1-\gamma}\Phi\left(\dfrac{(\overline{t}_1-\overline{t}_2)-u}{\left(\dfrac{\sqrt{n_2}s_1+\sqrt{n_1}s_2}{\sqrt{n_1n_2}}\right)}\right) & \text{if } (\overline{t}_1-\overline{t}_2)-\left(\dfrac{\sqrt{n_2}s_1+\sqrt{n_1}s_2}{\sqrt{n_1n_2}}\right)\Phi^{-1}\left(1-\dfrac{\gamma}{2}\right) \le u \le (\overline{t}_1-\overline{t}_2) \\ \dfrac{2-\gamma}{1-\gamma} - \dfrac{2}{1-\gamma}\Phi\left(\dfrac{u-(\overline{t}_1-\overline{t}_2)}{\left(\dfrac{\sqrt{n_2}s_1+\sqrt{n_1}s_2}{\sqrt{n_1n_2}}\right)}\right) & \text{if } (\overline{t}_1-\overline{t}_2) \le u \le (\overline{t}_1-\overline{t}_2)+\left(\dfrac{\sqrt{n_2}s_1+\sqrt{n_1}s_2}{\sqrt{n_1n_2}}\right)\Phi^{-1}\left(1-\dfrac{\gamma}{2}\right) \end{cases} \quad (20)$$

where $\gamma \in [0,1)$.

5.3 Fuzzy multiplication

Let $^{\alpha}\mathrm{M}_1 = \left[\underline{\mathrm{M}}_{1,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\right]$, $^{\alpha}\mathrm{M}_2 = \left[\underline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{2,\alpha}\right]$ be again two fuzzy mean estimators obtained from two normally distributed random samples $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ respectively. Then the fuzzy multiplication of M_1 and M_2 denoted by $\mathrm{M} = \mathrm{M}_1(*)\mathrm{M}_2$ is according to Definition 2.7

$$\left[\underline{\mathrm{M}}_{\alpha}, \overline{\mathrm{M}}_{\alpha}\right] = \left[\underline{\mathrm{M}}_{1,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\right](*)\left[\underline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{2,\alpha}\right] =$$
$$\Big[\min\left(\underline{\mathrm{M}}_{1,\alpha}\cdot\underline{\mathrm{M}}_{2,\alpha}, \underline{\mathrm{M}}_{1,\alpha}\cdot\overline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\cdot\underline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\cdot\overline{\mathrm{M}}_{2,\alpha}\right),$$
$$\max\left(\underline{\mathrm{M}}_{1,\alpha}\cdot\underline{\mathrm{M}}_{2,\alpha}, \underline{\mathrm{M}}_{1,\alpha}\cdot\overline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\cdot\underline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\cdot\overline{\mathrm{M}}_{2,\alpha}\right)\Big]$$

for each $\alpha \in (0,1]$.

It can be seen above that four multiplications on α-cuts intervals are required; therefore four membership functions also have to be combined. First, let us examine the case were the above intervals are in $\Re_+$, the non negative real line, then the multiplication formula gets simplified to

$$\left[\underline{\mathrm{M}}_{\alpha}, \overline{\mathrm{M}}_{\alpha}\right] = \left[\underline{\mathrm{M}}_{1,\alpha}\cdot\underline{\mathrm{M}}_{2,\alpha}, \overline{\mathrm{M}}_{1,\alpha}\cdot\overline{\mathrm{M}}_{2,\alpha}\right], \alpha \in (0,1].$$

By Theorem 4.1, the α-cuts of this fuzzy operation are

$$^{2g(\beta)}\mathrm{M}=\left[\left(\bar{t}_1-z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)\left(\bar{t}_2-z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right),\left(\bar{t}_1+z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)\left(\bar{t}_2+z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)\right],\beta\in[0,1]. \quad (21)$$

To find its membership function, let for the left side of the α-cut interval (21) be

$$u=\left(\bar{t}_1-z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)\left(\bar{t}_2-z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)$$

then

$$z_{l,g(\beta)}=\frac{\sqrt{n_1}s_2\bar{t}_1+\sqrt{n_2}s_1\bar{t}_2\pm\sqrt{n_1s_2^2\bar{t}_1^2+n_2s_1^2\bar{t}_2^2+2\sqrt{n_1}\sqrt{n_2}s_1s_2(2u-\bar{t}_1\bar{t}_2)}}{2s_1s_2} \quad (22)$$

and for the right side of (21)

$$z_{r,g(\beta)}=\frac{-\sqrt{n_1}s_2\bar{t}_1-\sqrt{n_2}s_1\bar{t}_2\pm\sqrt{n_1s_2^2\bar{t}_1^2+n_2s_1^2\bar{t}_2^2+2\sqrt{n_1}\sqrt{n_2}s_1s_2(2u-\bar{t}_1\bar{t}_2)}}{2s_1s_2} \quad (23)$$

But since $z_{g(\beta)}=\Phi^{-1}\left(1-g(\beta)\right)$ and $g(\beta)=\left(\frac{1}{2}-\frac{\gamma}{2}\right)\beta+\frac{\gamma}{2}$, it holds that

$$\beta=\frac{2}{1-\gamma}\Phi\left(-z_{g(\beta)}\right)+\frac{2-\gamma}{1-\gamma},\gamma\in[0,1). \quad (24)$$

By combining (22), (23), (24), considering the constraint $0\le\beta\le1$ and based on Theorem 3.2, we obtain the following fuzzy multiplication in $\Re_+$.

$$\mathrm{M}(u)=\begin{cases}\frac{2-\gamma}{1-\gamma}-\frac{2}{1-\gamma}\Phi\left(z_{l,g(\beta)}\right) & if\ \frac{(-Fs_1+\sqrt{n_1}\bar{t}_1)(-Fs_2+\sqrt{n_2}\bar{t}_2)}{\sqrt{n_1}\sqrt{n_2}}\le u\le(\bar{t}_1\cdot\bar{t}_2)\\ \frac{2-\gamma}{1-\gamma}-\frac{2}{1-\gamma}\Phi\left(z_{r,g(\beta)}\right) & if\ (\bar{t}_1\cdot\bar{t}_2)\le u\le\frac{(Fs_1+\sqrt{n_1}\bar{t}_1)(Fs_2+\sqrt{n_2}\bar{t}_2)}{\sqrt{n_1}\sqrt{n_2}}\end{cases} \quad (25)$$

where $F=\Phi^{-1}\left(1-\frac{\gamma}{2}\right),\gamma\in[0,1)$.

In the general case were the fuzzy multiplication is in $\Re$, the α-cuts are

$$^{2g(\beta)}\mathrm{M}=\left[\min\left(k_1(\beta),k_2(\beta),k_3(\beta),k_4(\beta)\right),\max\left(k_1(\beta),k_2(\beta),k_3(\beta),k_4(\beta)\right)\right],\beta\in[0,1] \quad (26)$$

where

$$k_1(\beta)=\left(\bar{t}_1-z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)\left(\bar{t}_2-z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right),\ k_2(\beta)=\left(\bar{t}_1-z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)\left(\bar{t}_2+z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)$$

$$k_3(\beta)=\left(\bar{t}_1+z_{l,g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)\left(\bar{t}_2-z_{r,g(\beta)}\frac{s_2}{\sqrt{n_2}}\right),\ k_4(\beta)=\left(\bar{t}_1+z_{l,g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)\left(\bar{t}_2+z_{r,g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)$$

To find the membership function, first we have to find $k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u),k_4^{-1}(u)$ as shown previously. For the left side of (26) it holds that $\underline{\mathrm{M}}^{-1}(u)=\max\left(k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u),k_4^{-1}(u)\right)$. Respectively, for the right side $\overline{\mathrm{M}}^{-1}(u)=\min\left(k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u),k_4^{-1}(u)\right)$. Therefore, based on Theorem 3.2, the membership function of the fuzzy multiplication in $\Re$ turns out to be

$$\mathrm{M}(u)=\begin{cases}\max\left(k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u),k_4^{-1}(u)\right) & if \;\; \dfrac{(-Fs_1+\sqrt{n_1}\bar{t}_1)(-Fs_2+\sqrt{n_2}\bar{t}_2)}{\sqrt{n_1}\sqrt{n_2}}\le u\le(\bar{t}_1\cdot\bar{t}_2)\\ \min\left(k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u),k_4^{-1}(u)\right) & if \;\; (\bar{t}_1\cdot\bar{t}_2)\le u\le\dfrac{(Fs_1+\sqrt{n_1}\bar{t}_1)(Fs_2+\sqrt{n_2}\bar{t}_2)}{\sqrt{n_1}\sqrt{n_2}}\end{cases} \quad (27)$$

where $F=\Phi^{-1}\left(1-\frac{\gamma}{2}\right),\gamma\in[0,1)$.

5.4 Fuzzy division

The fuzzy division of M_1 and M_2 denoted by $\mathrm{M}=\mathrm{M}_1(/)\mathrm{M}_2$ is obtained using the fuzzy multiplication as follows

$$\left[\underline{\mathrm{M}}_\alpha,\overline{\mathrm{M}}_\alpha\right]={}^\alpha\left[\mathrm{M}_1(/)\mathrm{M}_2\right]=\left[\underline{\mathrm{M}}_{1,\alpha},\overline{\mathrm{M}}_{1,\alpha}\right](*)\left[\frac{1}{\overline{\mathrm{M}}_{2,\alpha}},\frac{1}{\underline{\mathrm{M}}_{2,\alpha}}\right]=$$

$$=\left[\min\left(\frac{\underline{\mathrm{M}}_{1,\alpha}}{\overline{\mathrm{M}}_{2,\alpha}},\frac{\underline{\mathrm{M}}_{1,\alpha}}{\underline{\mathrm{M}}_{2,\alpha}},\frac{\overline{\mathrm{M}}_{1,\alpha}}{\overline{\mathrm{M}}_{2,\alpha}},\frac{\overline{\mathrm{M}}_{1,\alpha}}{\underline{\mathrm{M}}_{2,\alpha}}\right),\max\left(\frac{\underline{\mathrm{M}}_{1,\alpha}}{\overline{\mathrm{M}}_{2,\alpha}},\frac{\underline{\mathrm{M}}_{1,\alpha}}{\underline{\mathrm{M}}_{2,\alpha}},\frac{\overline{\mathrm{M}}_{1,\alpha}}{\overline{\mathrm{M}}_{2,\alpha}},\frac{\overline{\mathrm{M}}_{1,\alpha}}{\underline{\mathrm{M}}_{2,\alpha}}\right)\right]$$

for each $\alpha\in(0,1]$, provided $0\notin\left[\underline{\mathrm{M}}_{2,\alpha},\overline{\mathrm{M}}_{2,\alpha}\right]$.

In case these intervals are in $\Re_+$ this formula of fuzzy division is simplified to

$$\left[\underline{\mathrm{M}}_\alpha,\overline{\mathrm{M}}_\alpha\right]=\left[\frac{\underline{\mathrm{M}}_{1,\alpha}}{\overline{\mathrm{M}}_{2,\alpha}},\frac{\overline{\mathrm{M}}_{1,\alpha}}{\underline{\mathrm{M}}_{2,\alpha}}\right],\alpha\in(0,1].$$

By Theorem 4.1,

$${}^{2g(\beta)}\mathrm{M}=\left[\frac{\left(\bar{t}_1-z_{g(\beta)}\dfrac{s_1}{\sqrt{n_1}}\right)}{\left(\bar{t}_2+z_{g(\beta)}\dfrac{s_2}{\sqrt{n_2}}\right)},\frac{\left(\bar{t}_1+z_{g(\beta)}\dfrac{s_1}{\sqrt{n_1}}\right)}{\left(\bar{t}_2-z_{g(\beta)}\dfrac{s_2}{\sqrt{n_2}}\right)}\right],\beta\in[0,1] \quad (28)$$

and based on Theorem 3.2, the membership function is

$$\mathrm{M}(u)=\begin{cases}\dfrac{2-\gamma}{1-\gamma}-\dfrac{2}{1-\gamma}\Phi\left(z_{l,g(\beta)}\right) & if \;\; \dfrac{\sqrt{n_2}(-Fs_1+\sqrt{n_1}\bar{t}_1)}{\sqrt{n_1}(Fs_2+\sqrt{n_2}\bar{t}_2)}\le u\le\dfrac{\bar{t}_1}{\bar{t}_2}\\ \dfrac{2-\gamma}{1-\gamma}-\dfrac{2}{1-\gamma}\Phi\left(z_{r,g(\beta)}\right) & if \;\; \dfrac{\bar{t}_1}{\bar{t}_2}\le u\le-\dfrac{\sqrt{n_2}(Fs_1+\sqrt{n_1}\bar{t}_1)}{\sqrt{n_1}(Fs_2-\sqrt{n_2}\bar{t}_2)}\end{cases} \quad (29)$$

where

$$z_{l,g(\beta)}=\frac{\sqrt{n_1}\sqrt{n_2}(\bar{t}_1-u\bar{t}_2)}{\sqrt{n_2}s_1+u\sqrt{n_1}s_2},z_{r,g(\beta)}=\frac{\sqrt{n_1}\sqrt{n_2}(-\bar{t}_1+u\bar{t}_2)}{\sqrt{n_2}s_1+u\sqrt{n_1}s_2},F=\Phi^{-1}\left(1-\frac{\gamma}{2}\right),\gamma\in[0,1).$$

In the general case of the fuzzy division in $\Re$, the α-cuts are

$${}^{2g(\beta)}\mathrm{M}=\left[\min\left(k_1(\beta),k_2(\beta),k_3(\beta),k_4(\beta)\right),\max\left(k_1(\beta),k_2(\beta),k_3(\beta),k_4(\beta)\right)\right],\beta\in[0,1] \quad (30)$$

and the membership function turns out to be

$$\mathrm{M}(u)=\begin{cases}\max\left(k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u),k_4^{-1}(u)\right) & if\ \dfrac{\sqrt{n_2}(-Fs_1+\sqrt{n_1}\bar{t}_1)}{\sqrt{n_1}(Fs_2+\sqrt{n_2}\bar{t}_2)}\le u\le\dfrac{\bar{t}_1}{\bar{t}_2}\\ \min\left(k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u),k_4^{-1}(u)\right) & if\ \dfrac{\bar{t}_1}{\bar{t}_2}\le u\le\dfrac{(Fs_1+\sqrt{n_1}\bar{t}_1)(Fs_2+\sqrt{n_2}\bar{t}_2)}{\sqrt{n_1}\sqrt{n_2}}\end{cases} \tag{31}$$

where

$$k_1(\beta)=\frac{\left(\bar{t}_1-z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)}{\left(\bar{t}_2-z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)},k_2(\beta)=\frac{\left(\bar{t}_1-z_{g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)}{\left(\bar{t}_2+z_{g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)},k_3(\beta)=\frac{\left(\bar{t}_1+z_{l,g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)}{\left(\bar{t}_2-z_{r,g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)},k_4(\beta)=\frac{\left(\bar{t}_1+z_{l,g(\beta)}\frac{s_1}{\sqrt{n_1}}\right)}{\left(\bar{t}_2+z_{r,g(\beta)}\frac{s_2}{\sqrt{n_2}}\right)}$$

and $F=\Phi^{-1}\left(1-\frac{\gamma}{2}\right),\gamma\in[0,1)$.

6. Fuzzy ranking method using fuzzy estimators

Comparison of fuzzy numbers is an important component in fuzzy expert systems. Cheng (1998) proposed a method of ranking fuzzy numbers using the coefficient of variation, a statistical measure of dispersion, which is the ratio of standard deviation to the mean. Here, we propose the statistical measure of variance-to-the-mean ratio (VMR), defined as

$$VMR=\frac{\sigma^2}{\mu},\mu\neq 0\,. \tag{32}$$

but instead we consider σ^2 and μ to be fuzzy estimators, as defined in Section 4, to produce a fuzzy VMR. We show that this fuzzy measure has a well-defined membership function, therefore it can be used for fuzzy comparisons.

First we find the α-cuts of the fuzzy VMR. Using Nguyen's Proposition (1978), we have

$$^{a}VMR=\frac{^{a}\sigma^2}{^{a}\mu},\ a\in[0,1],0\notin{}^{\alpha}\mu\,. \tag{33}$$

According Definition 2.4, and the realization of the fuzzy division operation on the α-cuts, we obtain

$$[VMR_L,VMR_R]=[\sigma_L^2,\sigma_R^2]/[\mu_L,\mu_R]=\left[\frac{\sigma_L^2}{\mu_R},\frac{\sigma_R^2}{\mu_L}\right],\ 0\notin[\mu_L,\mu_R]\,. \tag{34}$$

Let now $X_1,X_2,...,X_n$ be a normally distributed random sample and $x_1,x_2,...x_n$ be sample values assumed by the sample. Since we can estimate the α-cuts of the fuzzy σ^2 and μ from this sample, by Theorems 3.1 and 3.3, (34) becomes

$$[VMR_L, VMR_R] = \left[\frac{\dfrac{s^2}{1+z_{g(\beta)}\sqrt{\dfrac{2}{n-1}}}}{\overline{x}+z_{g(\beta)}\dfrac{s}{\sqrt{n}}}, \frac{\dfrac{s^2}{1-z_{g(\beta)}\sqrt{\dfrac{2}{n-1}}}}{\overline{x}-z_{g(\beta)}\dfrac{s}{\sqrt{n}}} \right] \tag{35}$$

were $\overline{x}, s_1, n_1$ are the sample mean, sample standard deviation and sample size. To find formula of the fuzzy VMR, beginning with the left side of (35), we find

$$z_{l,g(\beta)} = -\frac{1}{2\sqrt{2}\sqrt{\frac{1}{-1+n}}su}(su+\sqrt{2}\sqrt{\frac{1}{-1+n}}\sqrt{n}u\overline{x} \pm \sqrt{u(s^2(4\sqrt{2}\sqrt{\frac{1}{-1+n}}\sqrt{n}s+u)-2\sqrt{2}\sqrt{\frac{1}{-1+n}}\sqrt{n}su\overline{x}+\frac{2nu\overline{x}^2}{-1+n}}))$$

$$\beta = \frac{2}{1-\gamma}\Phi\left(-z_{l,g(\beta)}\right)+\frac{2-\gamma}{1-\gamma} \tag{36}$$

From the right side we find

$$z_{r,g(\beta)} = \frac{1}{2\sqrt{2}\sqrt{\frac{1}{-1+n}}su}(su+\sqrt{2}\sqrt{\frac{1}{-1+n}}\sqrt{n}u\overline{x} \pm \sqrt{u(s^2(4\sqrt{2}\sqrt{\frac{1}{-1+n}}\sqrt{n}s+u)-2\sqrt{2}\sqrt{\frac{1}{-1+n}}\sqrt{n}su\overline{x}+\frac{2nu\overline{x}^2}{-1+n}}))$$

$$\beta = \frac{2}{1-\gamma}\Phi\left(-z_{r,g(\beta)}\right)+\frac{2-\gamma}{1-\gamma}. \tag{37}$$

Moreover, since $0 \le \beta \le 1$, (36) and (37) have to abide the following restrictions:

$$\frac{(-1+\sqrt{2}F\sqrt{\frac{1}{-1+n}})(-1+n)\sqrt{n}s^2}{(1+2F^2-n)(Fs+\sqrt{n}\overline{x})} \le u \le \frac{s^2}{\overline{x}} \tag{38}$$

$$\frac{s^2}{\overline{x}} \le u \le \frac{(1+\sqrt{2}F\sqrt{\frac{1}{-1+n}})(-1+n)\sqrt{n}s^2}{(1+2F^2-n)(Fs-\sqrt{n}\overline{x})} \tag{39}$$

By combining (36),(37),(38),(39) and based on Theorem 3.2, the membership function of the fuzzy VMR turns out to be

$$VMR(u) = \begin{cases} \frac{2-\gamma}{1-\gamma}-\frac{2}{1-\gamma}\Phi\left(z_{l,g(\beta)}\right) & if \quad \frac{(-1+\sqrt{2}F\sqrt{\frac{1}{-1+n}})(-1+n)\sqrt{n}s^2}{(1+2F^2-n)(Fs+\sqrt{n}\overline{x})} \le u \le \frac{s^2}{\overline{x}} \\ \frac{2-\gamma}{1-\gamma}-\frac{2}{1-\gamma}\Phi\left(z_{r,g(\beta)}\right) & if \quad \frac{s^2}{\overline{x}} \le u \le \frac{(1+\sqrt{2}F\sqrt{\frac{1}{-1+n}})(-1+n)\sqrt{n}s^2}{(1+2F^2-n)(Fs-\sqrt{n}\overline{x})} \end{cases} \tag{40}$$

where $F = \Phi^{-1}\left(1-\frac{\beta}{2}\right), \gamma \in [0,1)$.

Having found the fuzzy *VMR* for two fuzzy numbers, a comparison between them is now feasible. The fuzzy number with the smaller *VMR* value is ranked higher. We can implement this as a classical relation

$$Def\left(VMR_1\right) \oplus Def\left(VMR_1\right), \oplus = \{<,>,\leq,\geq,=\} \quad (41)$$

where $Def\left(.\right)$ denotes a defuzzification function that maps the fuzzy *VMR* onto a single discrete (crisp) real value. Table 1 displays all the possible cases.

Table 1. Fuzzy comparisons of two numbers estimated from statistical samples.

Crisp Comparison	Fuzzy Comparison
X < Y	$Def\left(VMR_X\right) < Def\left(VMR_Y\right)$
X > Y	$Def\left(VMR_X\right) > Def\left(VMR_Y\right)$
X ≤ Y	$Def\left(VMR_X\right) \leq Def\left(VMR_Y\right)$
X ≥ Y	$Def\left(VMR_X\right) \geq Def\left(VMR_Y\right)$

7. A Prolog implementation

Considering the formulae presented earlier to construct fuzzy numbers, to perform fuzzy arithmetic operations and to compare fuzzy numbers from statistical data, we proceed in this section to the use of fuzzy estimators in Prolog.

Prolog has two desirable features for performing computational tasks: (a) a procedural interpretation of its semantics (van Emden and Kowalski 1976) and (b) its ability to define custom arithmetic functions. According to (a), the left-hand side (called also head) of a Horn clause can be the name of a procedure and the right hand side (body) of the clause the operations to be performed. Thus, Prolog predicates are suitable as fuzzy arithmetic functions. By convention, the last term of such a Prolog predicate will be the output of a fuzzy arithmetic function and all the left terms will be the parameters that specify the fuzzy estimator. Tables 2 and 3 gather the parameters of fuzzy estimators' and the corresponding formulae of Sections 4 and 5.

Table 2. Fuzzy estimators' parameters

Fuzzy Estimator	Size of sample	Specifying parameters	Membership functions	α-cuts
Fuzzy μ	Large	μ, s, n, γ	(4)	(5)
Fuzzy μ	Small	μ, s, n, γ	(4) as modified by Remark 4.1	(5) as modified by Remark 4.1
Fuzzy σ^2	Small	s, n, γ	(10)	(11)
Fuzzy σ^2	Large	s, n, γ	(12)	(13)

Table 3. Fuzzy operations' parameters

Fuzzy Operation	Domain	Specifying parameters	Membership functions	α-cuts
Fuzzy $\mu_1 + \mu_2$	$\Re$	$\overline{t}_1, s_1, n_1, \overline{t}_2, s_2, n_2, \gamma$	(18)	(14)
Fuzzy $\mu_1 - \mu_2$	$\Re$	$\overline{t}_1, s_1, n_1, \overline{t}_2, s_2, n_2, \gamma$	(20)	(19)
Fuzzy $\mu_1 \cdot \mu_2$	$\Re_+$	$\overline{t}_1, s_1, n_1, \overline{t}_2, s_2, n_2, \gamma$	(25)	(21)
Fuzzy $\mu_1 \cdot \mu_2$	$\Re$	$\overline{t}_1, s_1, n_1, \overline{t}_2, s_2, n_2, \gamma$	(27)	(26)
Fuzzy μ_1 / μ_2	$\Re_+$	$\overline{t}_1, s_1, n_1, \overline{t}_2, s_2, n_2, \gamma$	(29)	(28)
Fuzzy μ_1 / μ_2	$\Re$	$\overline{t}_1, s_1, n_1, \overline{t}_2, s_2, n_2, \gamma$	(31)	(30)

7.1 Example 1

Our research includes the development of Prolog compilers which makes it easier to encapsulate fuzzy estimators, but hereinafter we present a Prolog implementation to construct fuzzy estimators to the knowledge engineer who has not access to compiler source code to modify the compiler its self. The syntax and semantics of Prolog (Roberts 1987) have been used. For a detailed exposure on Prolog's inference mechanism, we refer to Clocksin and Mellish (2003).

Prolog example that implements the fuzzy μ estimator.

```
domains
  m,s,gamma,r = real
  n = integer
predicates
  CDF(r,r,r,r)    #Cumulative Distribution Function of the
                  #standard normal distribution
  RCDF(r,r,r,r)   #Reverse CDF

  fuzzymean(r,m,s,n,gamma,r)   #Fuzzy estimator for the mean
                               #of the normal distribution
clauses
  CDF(Istart,Iend,Res0,Res):-abs(Istart-Iend)<=0.001,Res=Res0,!.
  CDF(Istart,Iend,Res0,Res) :-
    NewIstart=Istart+0.001,
    NewRes0=Res0+1/sqrt(2*3.14159265)*exp(-
Istart*Istart/2)*0.001,
    CDF(NewIstart,Iend,NewRes0,Res).

  RCDF(Istart,Iend,Res0,Res):-abs(Res0-Iend)<=0.001,Res=Istart,!.
  RCDF(Istart,Iend,Res0,Res) :-
     NewIstart=Istart+0.001,
     NewRes0=Res0+1/sqrt(2*3.14159265)*exp(-
Istart*Istart/2)*0.001,
     RCDF(NewIstart,Iend,NewRes0,Res).

 #left hand side of formula (4)
```

```
  fuzzymean(X,M,S,N,Gamma,Res) :-
     G = 1-Gamma/2,RCDF(-5,G,0,OutRF),
     M-(S/sqrt(N))*OutRF<=X,X<=M,
     Z = (M-X)/(S/sqrt(N)),CDF(-5,Z,0,OutF),
     Res = (2-Gamma)/(1-Gamma)-(2/(1-Gamma))*OutF.

 #right hand side of formula (4)
  fuzzymean(X,M,S,N,Gamma,Res) :-
     G = 1-Gamma/2,RCDF(-5,G,0,OutRF),
     M<X,X<=M+(S/sqrt(N))*OutRF,
     Z = (X-M)/(S/sqrt(N)),CDF(-5,Z,0,OutF),
     Res = (2-Gamma)/(1-Gamma)-(2/(1-Gamma))*OutF.
goal
  fuzzymean(39.7,40,4,100,0.1,Out)
```

```
The answer is:
Out = 0.3928
```

The above example deals with the fuzzy μ estimator of the normal distribution, in the sense that given an input value, the membership value of the fuzzy estimator is returned. As it can be seen, a fuzzy predicate function `fuzzymean (r,m,s,n,gamma,r)` has been introduced. Notice that two clauses instead of one with the same `fuzzymean` predicate name are required for the calculations of the left and the right hand side of the formula. Herein, CDF is the cumulative distribution function of the standard normal distribution and RCDF is the reverse CDF function. The above example is the prototype for any fuzzy estimators or fuzzy operations given their membership functions, such as those of columns 4 in Tables 2 and 3.

7.2 Example 2

An alternative implementation scheme is based on the α-cuts representation of fuzzy estimators. Using Prolog:

Step 1. A fuzzy domain type for fuzzy estimators is defined.

Step 2. A fuzzy predicate function that contains fuzzy estimators as terms of the
above fuzzy domain is declared for each binary operation.

Step 3. The result of an operation is stored in an output variable which is either exported or used in subsequent operations.

The following Prolog program finds the fuzzy average of two fuzzy mean estimators.

Example of a Prolog schema that implements fuzzy operations

```
1. domains
2.  a,m,s,gamma = real
3.  n = integer
4.  fuzzy = fuzzymean(m,s,n,gamma);
            fuzzyadd(m,s,n,m,s,n,gamma);
            singleton(a)
5. predicates
6.  fuzzyadd(fuzzy, fuzzy, fuzzy)
7.  fuzzydiv(fuzzy, fuzzy, fuzzy)
8.  fuzzyaverage(fuzzy, fuzzy, fuzzy)
9. clauses
10. fuzzyadd(X, Y, Z) :- Z = X f+ Y.
11. fuzzydiv(X, Y, Z) :- Z = X f/ Y.
12. fuzzyaverage(X, Y, Z) :-
     fuzzyadd(X,Y,W),fuzzydiv(W,(singleton (2)),Z).
13.goal
14. fuzzyaverage (fuzzymean(12.53,0.62,113,0.90]),
    (fuzzymean(17.23,2.18,37,0.90)), Out)
```

When the program executes, Prolog will search the list of clauses for a possible matching between the goal `fuzzyaverage` and the head of each clause. In the first search cycle only the third clause matches the goal. Then, the free variables of this clause will be bounded with the fuzzy terms of the goal and its antecedent fuzzy predicate functions will become sub-goals for Prolog to execute them. The output variables Z, W and Out are instantiated with the results of the fuzzy operations.

For the reader who is into compiler building, a modification of the compiler is required to deal with the representation of fuzzy numbers using the set of values $\left[A(\alpha),B(\alpha)\right]$, $\alpha \in (0,1]$, as well as the assignment of these fuzzy numbers to variables during terms' unification phase. To perform the fuzzy arithmetic operations (f+,f-,f*,f/) one can use the α-cuts intervals of column 5 in Table 3.

8. Summary

In this paper fuzzy numbers are constructed from statistical data, extending current approaches of fuzzy estimation to ensure compact support for fuzzy estimators, depending on a parameter $\gamma \in [0,1)$. This view enables interval operations to be directly extended to membership functions which can be implemented seamlessly in Prolog. When the statistical samples follow the normal distribution, the formulae of the membership functions have been presented. Further research is needed to deal with fuzzy estimators of other types of statistical distributions.

References

[1] B.G., Buchanan and E.H.Shortliffe, Rule-Based Expert Systems - The MYCIN Experiments of the Stanford Heuristic Programming Project. Addison-Wesley, 1984.

[2] J.J. Buckley, Fuzzy statistics: hypothesis testing, Soft Computing, 9 (2005), 512-518.

[3] J.J. Buckley, W. Siler and D. Tucker, A fuzzy expert system. Fuzzy Sets and Systems, 20(1), (1986), 1-16.

[4] C. Cheng, A new approach for ranking fuzzy numbers by distance method. Fuzzy Sets and Systems, 95(3), (1998), 307-317.

[5] W. Clocksin, and C. Mellish, Programming in PROLOG. Springer-Verlag, 2003.

[6] D. Dubois and H. Prade, Operations on fuzzy numbers. International Journal of Systems Science, 9(6), (1978), 613-626.

[7] A. Falsafain, S.M. Taheri and M. Mashinchi, Fuzzy estimation of parameters in statistical models. Intern. Journal of Mathematics Sciences, 2(2), (2008), 79-85.

[8] D.P. Filev and R.R. Yager, Operations on fuzzy numbers via fuzzy reasoning. Fuzzy Sets and Systems, 91(2), (1997), 137-142.

[9] J. Fodor and B. Bede, Arithmetics with Fuzzy Numbers: A Comparative Overview. In: Proc. of 4th Slovakian-Hungarian Joint Symposium on Applied Machine Intelligence. Herľany, 2006.

[10] M. Gallanti, G. Guida, L. Spampinato and A. Stefanini, Representing procedural knowledge in expert systems: An application to process control. In: Proceedings, 9th Intern. Joint Conf. on Artificial Intelligence, (1985), 345–352.

[11] C.J. Hinde, Fuzzy Prolog. Inter. Journal of Man-Machine Studies, 24 (1986), 569-595.

[12] T. Hong and C. Lee, Induction of fuzzy rules and membership functions from training examples. Fuzzy Sets and Systems, 84(1), (1996), 33-47.

[13] G.J. Klir and B. Yuan, Fuzzy sets and fuzzy logic: theory and applications. Englewood Cliffs, NJ: Prentice Hall, 1995.

[14] V. Kreinovich, C. Quintana and L. Reznik, Gaussian membership functions are most adequate in representing uncertainty in measurements. Proceedings of NAFIPS, 92(12), (2003), 15–17.

[15] K.S. Leung and W. Lam, A fuzzy expert system shell using both exact and inexact reasoning. Journal of Automated Reasoning, 5(2), (1989), 207–233.

[16] T.P. Martin, J.F. Baldwin and B.W. Pilsworth, The implementation of fprolog—a fuzzy prolog interpreter. Fuzzy Sets and Systems, 23(1), (1987), 119–129.

[17] H.T. Nguyen, A Note on the Extension Principle for Fuzzy Sets, Journal of Mathematical Analysis and Applications 64 (1978), 369-380.

[18] B. Orchard, FuzzyCLIPS Version 6.10 d User's Guide, 2004.
[19] J. Pan, G.N. Desouza and A.C. Kak, FuzzyShell: a large-scale expert system shell using fuzzy logic for uncertainty reasoning. IEEE Transactions on Fuzzy Systems, 6(4), (1998), 563-581.
[20] R. Roberts, The power of Turbo Prolog: the natural language of artificial intelligence. TAB Books Inc, 1987.
[21] W. Siler, FLOPS: a fuzzy expert system shell. Preprints of Second IFSA Congress II, (1987) 848–850.
[22] M. Tsutomu, K. Hiroshi, F. Satoru, T. Shun'ichi, and U. Motohide, Fuzzy expert system shell LIFE FEShell-working environment. In: Proceedings, 2nd International Symposium on Uncertainty Modeling and Analysis, (1993), 153–160.
[23] M.H. van Emden and R.A. Kowalski, The Semantics of Predicate Logic as a Programming Language. Journal of the ACM, 23(4), (1976), 733-742.
[24] D. Wagman, M. Schneider and E. Schnaider, On the use of interval mathematics in fuzzy expert systems. International Journal of Intelligent Systems, 9 (1994), 241–259.
[25] C. Wu and B. Zhang, Embedding problem of noncompact fuzzy number space $E^{\sim}$ (II). Fuzzy Sets and Systems, 110 (2000), 135-142.
[26] L.A. Zadeh, The role of fuzzy logic in the management of uncertainty in expert systems. Fuzzy Sets and Systems, 11 (1983), 199-227.

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